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Skew braces and Rota–Baxter operators on semi-direct products

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Abstract. This paper examines the connections between (relative) Rota–Baxter groups, skew left braces, and enlargements of these structures on naturally associated semi-direct products. Given a skew left brace, we define a new skew left brace, referred to as its square, on the natural semi-direct product of its additive and multiplicative groups. Further, the square construction is distinct from the previously known double construction arising as a special case of matched pairs of skew braces. This provides a method to construct a new bijective, non-degenerate solution to the Yang–Baxter equation from an existing solution arising from a skew left brace. We show that the square construction is functorial and integrates naturally into both the cohomological and extension-theoretic frameworks for (relative) Rota–Baxter groups and skew left braces. Furthermore, we provide a sufficient condition under which two isoclinic skew left braces yield isoclinic squares.

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Contents

1 Introduction

The program for classifying set-theoretic solutions to the Yang–Baxter equation was proposed by Drinfel'd in the 1990s [15]. A key result by Guarnieri and Vendramin [17] established that every skew left brace yields a non-degenerate bijective set-theoretic solution to the Yang–Baxter equation. Recently, Guo, Lang, and Sheng [18] introduced Rota–Baxter operators on Lie groups as a means of producing such operators on the corresponding Lie algebras. The study of Rota–Baxter operators on more general groups, beyond the Lie group setting, was undertaken by Bardakov and Gubarev [2,3], who proved that every Rota–Baxter operator on a group induces a skew left brace structure on that group. This result links Rota–Baxter operators on groups directly to Drinfel'd's program. At the same time, Caranti and Stefanello [8] gave a cohomological characterization of when a skew left brace can be induced from a Rota–Baxter group. The idea of Rota–Baxter operators was further generalized by Jiang, Sheng, and Zhu [19] to relative Rota–Baxter operators on Lie groups. Building on this, it was proved in [23] that every skew left brace is induced by a relative Rota–Baxter group, and that there is an isomorphism between the categories of bijective relative Rota–Baxter groups and skew left braces. In [1], Bai et al. further connected relative Rota–Baxter groups to Butcher groups, post-groups, and pre-groups. As another generalization, Catino et al. [10, 12] introduced Rota–Baxter operators on Clifford semigroups and established a correspondence between such operators and weak braces, the latter being known to produce set-theoretic solutions to the Yang–Baxter equation that may be degenerate.

A cohomology theory for relative Rota–Baxter operators on Lie groups was introduced in [19]. However, this theory is not applicable to abstract groups, as it is defined in terms of the cohomology of the descendent group with coefficients in the Lie algebra of the acting Lie group, viewed as a module over the descendent group. To address this limitation, a cohomology and extension theory for relative Rota–Baxter groups was developed in [6], enabling the classification of such extensions via the second cohomology group. The relationship between the cohomology of relative Rota–Baxter groups and that of their associated skew left braces was also investigated. These constructions were subsequently applied in [5] to define the Schur multiplier and Schur covers of relative Rota–Baxter groups, and their connections to the corresponding constructions for skew left braces were explored.

Given the central role of skew left braces in classifying set-theoretic solutions to the Yang–Baxter equation, developing new methods for their construction is of fundamental importance. In this paper, we define a novel construction, called the square of a skew left brace, which produces a skew left brace from a given one. This construction is defined on the natural semi-direct product of the additive and multiplicative groups of the given skew left brace. The idea is motivated by an observation by Bardakov et al. [4] that every relative Rota–Baxter group gives rise to a Rota–Baxter operator on the natural semi-direct product of its constituent groups. We observe that our square construction is distinct from the previously known double construction of a skew left brace due to Smoktunowicz and Vendramin [25]. This provides a method to construct a new bijective, non-degenerate solution to the Yang–Baxter equation from an existing solution arising from a skew left brace. Further, we show that the square construction is functorial and naturally integrates into both the cohomological and extension-theoretic frameworks for (relative) Rota–Baxter groups and skew left braces.

The paper is organised as follows. In Section 2, we recall some necessary background from the extension and cohomology theory of skew left braces, Rota–Baxter groups and relative Rota–Baxter groups. In Section 3, we prove that the construction of the Rota–Baxter group on the associated semi-direct product arising from a given relative Rota–Baxter group is functorial (Proposition 3.2), and yields a homomorphism from the second cohomology of the relative Rota–Baxter group to that of the induced Rota–Baxter group (Proposition 3.4). In Section 4, we give the construction of the square of a skew left brace, and illustrate how it differs from its double. We prove that the construction of the square of a skew left brace is functorial (Proposition 4.8), and that there is a homomorphism from the second cohomology of the given skew left brace to that of its square (Proposition 4.10). In Section 5, we prove that, for each skew left brace, the homomorphisms of cohomology groups defined in the preceding sections give rise to a commutative diagram of cohomology groups of (relative) Rota–Baxter groups, skew left braces, and their squares (Theorem 5.1). Finally, in Section 6, we provide a sufficient condition under which two isoclinic skew left braces yield isoclinic squares (Theorem 6.6).

2 Preliminaries

In this section, we recall some essential results on abelian extensions of skew left braces and (relative) Rota–Baxter groups.

2.1 Abelian extensions of skew left braces

We begin by recalling some basic ideas about abelian extensions of skew left braces.

Definition 2.1. A skew left brace is a triple $(H, \cdot, \circ)$, where $(H, \cdot)$ and $(H, \circ)$ are groups such that

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)$$

for all $a, b, c \in H$, where a^{-1} denotes the inverse of a in $(H, \cdot)$. The groups $(H, \cdot)$ and $(H, \circ)$ are called the additive and the multiplicative groups of the skew left brace $(H, \cdot, \circ)$, respectively. When the additive group is abelian, then $(H, \cdot, \circ)$ is simply called a left brace.

Any group can be viewed as a skew left brace with the same additive and the multiplicative group, called the *trivial skew left brace*. Given a skew left brace $(H, \cdot, \circ)$, there is an associated action $\lambda^H : (H, \circ) \to \mathrm{Aut}(H, \cdot)$, which is a group homomorphism defined by

$$\lambda^H_a(b) = a^{-1} \cdot (a \circ b)$$

for all $a, b \in H$. The associated action plays a key role in understanding skew left braces.

Notation. *For simplicity in notation, we make the following conventions:*

1. *We denote the additive and the multiplicative groups of a skew left brace* $(H, \cdot, \circ)$ *by* $H^{(\cdot)}$ *and* $H^{(\circ)}$*, respectively.*
2. *When there is no ambiguity, we write the group operation in* $H^{(\cdot)}$ *simply as* ab*.*
3. *We denote the inverse of an element* g *in* $H^{(\cdot)}$ *by* g^{-1}*, and the inverse of* g *in* $H^{(\circ)}$ *by* $g^{\dagger}$*.*
4. *When the context is clear, we will not differentiate between the notations for the additive and the multiplicative groups of different skew left braces.*
5. *We will sometimes denote the image* $\phi(x)$ *of a homomorphism* ϕ *by* ϕ_x*.*
6. *We denote by* $\mathbf{1}$ *the trivial skew left brace for which both the underlying groups are trivial.*

Next, we introduce homomorphisms, ideals, and extensions of skew left braces.

Definition 2.2. Let $(H, \cdot_H, \circ_H)$ and $(K, \cdot_K, \circ_K)$ be skew left braces. A map $f : H \to K$ is called a homomorphism of skew left braces if

$$f(a \cdot_H b) = f(a) \cdot_K f(b) \quad \text{and} \quad f(a \circ_H b) = f(a) \circ_K f(b)$$

for all $a, b \in H$.

Definition 2.3. Let $(H, \cdot, \circ)$ be a skew left brace. A normal subgroup I of both $(H, \cdot)$ and $(H, \circ)$ is said to be an ideal of $(H, \cdot, \circ)$ if $\lambda^H_h(I) \subseteq I$ for all $h \in H$.

Definition 2.4. Let $(I, \cdot)$ be an abelian group viewed as a trivial brace and $(M, \cdot, \circ)$ be a skew left brace. An (abelian) extension of $(M, \cdot, \circ)$ by $(I, \cdot)$ is a skew left brace $(E, \cdot, \circ)$ that fits into the sequence

$$\mathcal{E}: \qquad \mathbf{1} \to (I, \cdot) \xrightarrow{i} (E, \cdot, \circ) \xrightarrow{\pi} (M, \cdot, \circ) \to \mathbf{1}$$

where i and π are homomorphisms of skew left braces such that i is injective, π is surjective and $\mathrm{im}(i) = \ker(\pi)$.

For simplicity, we denote $i(y)$ by y. By a (normalised) set-theoretic section to $\mathcal{E}$, we mean a map $s : H \to E$ such that $\pi\, s = \mathrm{id}_H$ and $s(1) = 1$. The equivalence of extensions of skew left braces is defined analogously to the equivalence of extensions of groups and other algebraic structures. We recall essential results on extensions of skew left braces by abelian groups from [24] (see also [20]).

Let $(M, \cdot, \circ)$ be a skew left brace and

$$\mathcal{E}: \qquad \mathbf{1} \longrightarrow (I, \cdot) \xrightarrow{\ i\ } (E, \cdot, \circ) \xrightarrow{\ \pi\ } (M, \cdot, \circ) \longrightarrow \mathbf{1}$$

be an extension of $(M, \cdot, \circ)$ by the trivial brace $(I, \cdot)$. Let $s : M \to E$ be a set-theoretic section to $\mathcal{E}$. We define maps $\xi, \epsilon : M^{(\circ)} \to \mathrm{Aut}(I)$ and $\zeta : M^{(\cdot)} \to \mathrm{Aut}(I)$ by

$$\begin{aligned} \xi_m(y) &= \lambda^E_{s(m)}(y), && (1)\\ \zeta_m(y) &= s(m)^{-1} \cdot y \cdot s(m) \ \text{ and} && (2)\\ \epsilon_m(y) &= s(m)^{\dagger} \circ y \circ s(m) && (3) \end{aligned}$$

for $m \in M$ and $y \in I$, where x^{-1} and $x^{\dagger}$ denote the inverse of x in $E^{(\cdot)}$ and $E^{(\circ)}$, respectively. It is not difficult to see that the map ξ is a homomorphism, whereas the maps ζ and ϵ are anti-homomorphisms. Furthermore, these maps are independent of the choice of the set-theoretic section [24, Proposition 3.4]. The triplet (ξ, ζ, ϵ) is called the *associated action* of the extension $\mathcal{E}$.

Next, recall the definition of the second cohomology group of a skew left brace $(M, \cdot, \circ)$ with coefficients in an abelian group $(I, \cdot)$ viewed as a trivial brace. Let $\xi : M^{(\circ)} \to \mathrm{Aut}(I)$ be a homomorphism and $\zeta : M^{(\cdot)} \to \mathrm{Aut}(I)$ and $\epsilon : M^{(\circ)} \to \mathrm{Aut}(I)$ be anti-homomorphisms satisfying the conditions

$$\begin{aligned} \xi_{m_1 \cdot m_2}(\epsilon_{m_1 \cdot m_2}(y))\, \zeta_{m_2}(y) &= \zeta_{m_2}(\xi_{m_1}(\epsilon_{m_1}(y)))\, \xi_{m_2}(\epsilon_{m_2}(y)) \text{and}\\ \zeta_{m_1^{-1} \cdot (m_1 \circ m_2)}(\xi_{m_1}(y)) &= \xi_{m_1}(\zeta_{m_2}(y)) \end{aligned}$$

for all $m_1, m_2 \in M$ and $y \in I$. Such a triplet (ξ, ζ, ϵ) is referred to as a *good triplet* of action of M on I.

Let $g, f : M \times M \to I$ be maps satisfying

$$g(m_2, m_3)\, g(m_1 \cdot m_2, m_3)^{-1}\, g(m_1, m_2 \cdot m_3)\, \zeta_{m_3}(g(m_1, m_2))^{-1} = 1, \qquad (4)$$

$$\xi_{m_1}(f(m_2,m_3))f(m_1\circ m_2,m_3)^{-1} \\ f(m_1,m_2\circ m_3)\,\xi_{m_1\circ m_2\circ m_3}\,(\epsilon_{m_3}(\xi^{-1}_{m_1\circ m_2}f(m_1,m_2)))^{-1}=1, \tag{5}$$

$$\xi_{m_1}(g(m_2,m_3))\zeta_{m_1\circ m_3}(g(m_1,m_1^{-1})) \\ \zeta_{m_1\circ m_3}(g(m_1\circ m_2,m_1^{-1}))^{-1}\,g((m_1\circ m_2)m_1^{-1},m_1\circ m_3)^{-1} \\ \zeta_{-m_1\cdot(m_1\circ m_3)}\,(f(m_1,m_2))^{-1}\,f(m_1,m_2\cdot m_3)\,f(m_1,m_3)^{-1}=1, \tag{6}$$

for all $m_1,m_2,m_3\in M$. Let $\mathrm{Z}^2_{SB}(M,I)$ be the set formed by pairs (g,f) of functions $g,f:M\times M\to I$ that satisfy (4), (5), (6) and vanish on degenerate tuples, and let $\mathrm{B}^2_{SB}(M,I)$ be the set of pairs $(g,f)\in\mathrm{Z}^2_{SB}(M,I)$ such that there exists a map $\theta:M\to I$ satisfying

$$\begin{aligned} g(m_1,m_2) &= \theta(m_1\cdot m_2)^{-1}\zeta_{m_2}(\theta(m_1))\theta(m_2)\text{and} && (7)\\ f(m_1,m_2) &= \theta(m_1\circ m_2)^{-1}\xi_{m_1\circ m_2}(\epsilon_{m_2}(\xi^{-1}_{m_1}(\theta(m_1))))\xi_{m_1}(\theta(m_2)) && (8)\end{aligned}$$

for all $m_1,m_2\in M$. Then the second cohomology group of $(M,\cdot,\circ)$ with coefficients in I corresponding to the given good triplet of actions (ξ,ζ,ϵ) is defined as

$$\mathrm{H}^2_{SB}(M,I)=\mathrm{Z}^2_{SB}(M,I)/\,\mathrm{B}^2_{SB}(M,I).$$

Let $\mathrm{Ext}_{(\xi,\zeta,\epsilon)}(M,I)$ denote the set of equivalence classes of those skew left brace extensions of M by I whose corresponding triplet of actions is (ξ,ζ,ϵ). Then the following result holds [24, Theorem A].

Theorem 2.5. *Let $(M,\cdot,\circ)$ be a skew left brace and $(I,\cdot)$ an abelian group viewed as a trivial brace. Then there is a bijection $\Upsilon:\mathrm{Ext}_{(\xi,\zeta,\epsilon)}(M,I)\to\mathrm{H}^2_{SB}(M,I)$ given by $\Upsilon([\mathcal{E}])=[\tau_1,\tau_2]$, where*

$$\begin{aligned} \tau_1(m_1,m_2) &= s(m_1\cdot m_2)^{-1}\cdot s(m_1)\cdot s(m_2)\ \textit{and} && (9)\\ \tau_2(m_1,m_2) &= s(m_1\circ m_2)^{-1}\cdot\big(s(m_1)\circ s(m_2)\big) && (10)\end{aligned}$$

for all $m_1,m_2\in M$ and s is a set-theoretic section to $\mathcal{E}$.

2.2 Abelian extensions of Rota–Baxter groups

Next, we recall some necessary results on Rota–Baxter groups from [14, Section 3, 4].

Definition 2.6. Let $(G,\cdot)$ be a group. A map $R:G\to G$ is called a Rota–Baxter operator of weight 1 on G if

$$R(x)\cdot R(y)=R\big(x\cdot R(x)\cdot y\cdot R(x)^{-1}\big),$$

for all $x,y\in G$. A group $(G,\cdot)$ equipped with a Rota–Baxter operator of weight 1 is called a Rota–Baxter group, and is denoted by the pair (G,R).

The following result connects Rota–Baxter groups to skew left braces [2, Proposition 3.1].

Proposition 2.7. *Let* $(G, \cdot)$ *be a group and* $R : G \to G$ *be a Rota–Baxter operator. If we define* $x \circ_R y = x \cdot R(x) \cdot y \cdot R(x)^{-1}$, *then* $(G, \cdot, \circ_R)$ *is a skew left brace.*

Note that, if the skew left brace $(G, \cdot, \circ_R)$ is induced by a Rota–Baxter operator $R : G \to G$, then the associated action is given by

$$\lambda^G_x(y) = R(x) \cdot y \cdot R(x)^{-1}, \tag{11}$$

which is simply the conjugation action by the image of R.

Definition 2.8. Let (G, R_G) and (H, R_H) be Rota–Baxter groups. A map $\phi : G \to H$ is called a homomorphism of Rota–Baxter groups if ϕ is a group homomorphism satisfying

$$\phi\, R_G = R_H\, \phi.$$

Definition 2.9. Let (I, R_I) and (H, R_H) be Rota–Baxter groups. A Rota–Baxter extension of (H, R_H) by (I, R_I) is a Rota–Baxter group (E, R_E) that fits into the sequence

$$\mathcal{E} : \qquad 1 \to (I, R_I) \overset{i}{\to} (E, R_E) \overset{\pi}{\to} (H, R_H) \to 1,$$

where i and π are homomorphisms of Rota–Baxter groups such that i is injective, π is surjective and $\mathrm{im}(i) = \ker(\pi)$.

For simplicity, we denote $i(y)$ by y, which implies that R_E restricted to I is R_I. The equivalence of extensions of Rota–Baxter groups is defined in a manner analogous to the standard notion of equivalence for extensions of groups and skew braces.

Remark 2.10. A direct check shows that an extension of Rota–Baxter groups induces an extension of induced skew left braces.

Let $\mathcal{E} : 1 \to (I, R_I) \overset{i}{\to} (E, R_E) \overset{\pi}{\to} (H, R_H) \to 1$ be a Rota–Baxter extension of (H, R_H) by (I, R_I), where I is an abelian group. By a (normalised) set-theoretic section to $\mathcal{E}$, we mean a map $s : H \to E$ such that $\pi\, s = \mathrm{id}_H$ and $s(1) = 1$. Let $\gamma : H \to \mathrm{Aut}(I)$ be the map defined by

$$\gamma_h(y) = s(h)^{-1} y s(h).$$

Note that γ is an anti-homomorphism, and is independent of the choice of a set-theoretic section. We call γ the *associated action* of the extension $\mathcal{E}$. It is not difficult to see that equivalent extensions of Rota–Baxter groups have the identical associated actions. For each $a \in E$, there exists unique $h \in H$ and $y \in I$ such that $a = s(h)y$. Hence, we can write

$$R_E\big(s(h)\big) = s\big(R_H(h)\big) y_h \tag{12}$$

for some unique $y_h \in I$. Consider the maps $\tau \colon H \times H \to I$ and $r \colon H \to I$ given by

$$\tau(h_1, h_2) = s(h_1 h_2)^{-1} s(h_1) s(h_2) \quad \text{and} \quad r(h) = y_h$$

for $h_1, h_2, h \in H$.

Definition 2.11. Let (H, R_H) be a Rota–Baxter group, let I be an abelian group, and let $R_I : I \to I$ be a group homomorphism. We say that (I, R_I) is a right (H, R_H)-module if I is a right H-module by an action $\gamma : H \to \mathrm{Aut}(I)$ and the condition

$$\gamma_{R_H(h)}\big(R_I(z)\big) = R_I\big(\gamma_{hR_H(h)}(z + R_I(z)) - \gamma_{R_H(h)}(R_I(z))\big)$$

holds for all $h \in H$ and $z \in I$.

Let (I, R_I) be an (H, R_H)-module by an action γ. Let $C^n(H, I)$ be the set of all maps $H^n \to I$ that vanish on all degenerate tuples. Define:

$$TC^1_{RB}(H, I) = C^1(H, I), \quad TC^2_{RB}(H, I) = C^2(H, I) \oplus C^1(H, I),$$
$$TC^3_{RB}(H, I) = C^3(H, I) \oplus C^2(H, I).$$

Define $\partial^1_{RB} : TC^1_{RB}(H, I) \to TC^2_{RB}(H, I)$ by

$$\partial^1_{RB}(\theta) = \big(\delta^1(\theta), -\Phi^1(\theta)\big),$$

where $\Phi^1(\theta)(h) = R_I\big(\gamma_{R_H(h)}(\theta(h))\big) - \theta\big(R_H(h)\big)$ and δ^1 is the standard 1-coboundary map defining the group cohomology of H with coefficients in I (see [7, Section 3.1]).

Similarly, define $\partial^2_{RB} : TC^2_{RB}(H, I) \to TC^3_{RB}(H, I)$ by

$$\partial^2_{RB}(f, g) = (\delta^2 f, \beta),$$

where

$$\begin{aligned}
\beta(h_1, h_2) &= \partial^1(g)(h_1, h_2) - R_I\big(\gamma_{R_H(h_2)}(\gamma_{h_2}(g(h_1)) - g(h_1))\big) - \Phi^2(f)(h_1, h_2),\\
\partial^1(g)(h_1, h_2) &= g(h_2) - g(h_1 \circ_{R_H} h_2) + \gamma_{R_H(h_2)}(g(h_1)),\\
\Phi^2(f)(h_1, h_2) &= f(R_H(h_1), R_H(h_2)) - R_I\Big(\gamma_{R_H(h_1 \circ_{R_H} h_2)}(f(h_1 R_H(h_1), h_2 R_H(h_1)^{-1}))\\
&\quad + \gamma_{h_2 R_H(h_1)^{-1}}(f(h_1, R_H(h_1))) + f(h_2, R_H(h_1)^{-1}) - f(R_H(h_1), R_H(h_1)^{-1})\Big),
\end{aligned}$$

and δ^2 is the standard 2-coboundary map defining the group cohomology of H with coefficients in I (see [7, Section 3.1]). Let

$$\mathrm{B}^2_{RB}(H, I) = \mathrm{im}(\partial^1_{RB}) \quad \text{and} \quad \mathrm{Z}^2_{RB}(H, I) = \ker(\partial^2_{RB}).$$

Then we define the second cohomology of (H, R_H) with coefficients in (I, R_I) by

$$\mathrm{H}^2_{RB}(H, I) = \mathrm{Z}^2_{RB}(H, I) / \mathrm{B}^2_{RB}(H, I).$$

With the preceding set-up, the following result holds [14, Theorem 4.4].

Theorem 2.12. *Let (H, R_H) be a Rota–Baxter group, (I, R_I) an $(H, R_H)-$module and $\mathrm{Ext}_\gamma(H, I)$ be the set of equivalence classes of all extensions of (H, R_H) by (I, R_I) inducing the action γ. Then the function $\Lambda : \mathrm{Ext}_\gamma(H, I) \to \mathrm{H}^2_{RB}(H, I)$ given by $\Lambda([\mathcal{E}]) = [\tau, r]$ is a bijection, such that*

$$\tau(h_1, h_2) = s(h_1 h_2)^{-1} s(h_1) s(h_2) \quad \textit{and} \quad r(h) = s\big(R_H(h)\big)^{-1} R_E\big(s(h)\big)$$

for all $h_1, h_2, h \in H$ and s is a set-theoretic section to $\mathcal{E}$.

2.3 Abelian extensions of relative Rota–Baxter groups

Finally, we recall essential results on extension theory of relative Rota–Baxter groups from [6].

Definition 2.13. A relative Rota–Baxter group is a quadruple (H, G, ϕ, R), where H and G are groups, $\phi : G \to \operatorname{Aut}(H)$ a group homomorphism and $R : H \to G$ is a map satisfying the condition

$$R(h_1)R(h_2) = R\big(h_1\phi_{R(h_1)}(h_2)\big),$$

for all $h_1, h_2 \in H$. The map R is referred to as the relative Rota–Baxter operator on H. The relative Rota–Baxter group (H, G, ϕ, R) is called trivial if $\phi : G \to \operatorname{Aut}(H)$ is the trivial homomorphism.

Remark 2.14. Let $\phi : G \to \operatorname{Aut}(G)$ be the adjoint action, that is, $\phi_y(x) = yxy^{-1}$ for $x, y \in G$. Then the relative Rota–Baxter group (G, G, ϕ, R) is simply the Rota–Baxter group (G, R).

The following result connects relative Rota–Baxter groups to skew left braces [23, Proposition 3.6].

Proposition 2.15. *If (H, G, ϕ, R) is a relative Rota–Baxter group, then $(H, \cdot, \circ_R)$ is a skew left brace, where $\cdot$ denotes the group operation on H and $\circ_R$ is defined by*

$$h_1 \circ_R h_2 = h_1 \cdot \phi_{R(h_1)}(h_2)$$

for $h_1, h_2 \in H$.

In the converse direction, it is known from [23, Theorem 3.10] that if $(H, \cdot, \circ)$ is a skew left brace, then $(H^{(\cdot)}, H^{(\circ)}, \lambda^H, \mathrm{id}_H)$ is a relative Rota–Baxter group.

Definition 2.16. Let (H, G, ϕ, R) and (K, L, φ, S) be two relative Rota–Baxter groups. A pair $(f_1, f_2) : (H, G, \phi, R) \to (K, L, \varphi, S)$ is called a homomorphism of relative Rota–Baxter groups if $f_1 : H \to K$ and $f_2 : G \to L$ are group homomorphisms such that

$$f_2\, R = S\, f_1 \quad \text{and} \quad f_1\, \phi_g = \varphi_{f_2(g)}\, f_1 \tag{13}$$

for all $g \in G$.

We write $\mathbf{1}$ to denote the trivial relative Rota–Baxter group for which both the underlying groups and the maps are trivial.

Definition 2.17. Let (K, L, α, S) and (A, B, β, T) be relative Rota–Baxter groups. An extension of (A, B, β, T) by (K, L, α, S) is a relative Rota–Baxter group (H, G, ϕ, R) that fits into the sequence

$$\mathcal{E} : \qquad \mathbf{1} \longrightarrow (K, L, \alpha, S) \overset{(i_1, i_2)}{\longrightarrow} (H, G, \phi, R) \overset{(\pi_1, \pi_2)}{\longrightarrow} (A, B, \beta, T) \longrightarrow \mathbf{1},$$

where (i_1, i_2) and (π_1, π_2) are homomorphisms of relative Rota–Baxter groups such that (i_1, i_2) is an embedding, (π_1, π_2) is an epimorphism of relative Rota–Baxter groups and $(\mathrm{im}(i_1), \mathrm{im}(i_2), \phi|, R|) = (\ker(\pi_1), \ker(\pi_2), \phi|, R|)$.

We say that $\mathcal{E}$ is an abelian extension if K and L are abelian groups and the relative Rota–Baxter group (K, L, α, S) is trivial.

Equivalence of extensions of relative Rota–Baxter groups is defined in the same way as for groups, skew braces and Rota–Baxter groups. Throughout the immediate discussion, $\mathcal{E}$ denotes the abelian extension

$$\mathbf{1} \longrightarrow (K, L, \alpha, S) \overset{(i_1,i_2)}{\longrightarrow} (H, G, \phi, R) \overset{(\pi_1,\pi_2)}{\longrightarrow} (A, B, \beta, T) \longrightarrow \mathbf{1}$$

of relative Rota–Baxter groups and (s_H, s_G) denotes a set-theoretic section to $\mathcal{E}$.

Proposition 2.18 ([6, p.11]). *Let $\mathcal{E}$ be an abelian extension of relative Rota–Baxter groups. Let $a \in A$, $b \in B$, $k \in K$, and $l \in L$. Then the following statements hold:*

1. *The action ϕ is characterised by the equation*

$$\phi_{s_G(b)l}\big(s_H(a)k\big) = s_H\big(\beta_b(a)\big)\, \rho(a, b)\, \phi_{s_G(b)}\big(f(l, a)k\big), \tag{14}$$

where $f : L \times A \to K$ is given by

$$f(l, a) = s_H(a)^{-1}\phi_l(s_H(a))$$

and $\rho : A \times B \to K$ is given by

$$\rho(a, b) = (s_H(\beta_b(a)))^{-1}\phi_{s_G(b)}(s_H(a)).$$

2. *The relative Rota–Baxter operator R is expressed as*

$$R\big(s_H(a)k\big) = s_G(T(a))\, \chi(a)\, S\big(\phi^{-1}_{s_G(T(a))}(k)\big), \tag{15}$$

where $\chi : A \to K$ is given by $\chi(a) = s_G(T(a))^{-1}R(s_H(a))$.

Lemma 2.19 ([6, Lemma 3.9]). *Let $\mathcal{E}$ be an abelian extension of relative Rota–Baxter groups. Then the following assertions hold:*

1. *The map $f : L \times A \longrightarrow K$ is independent of the choice of the section s_H.*
2. *$f(l_1 l_2, a) = f(l_1, a)f(l_2, a)$ for all $l_1, l_2 \in L$ and $a \in A$.*
3. *$f(l, a_1 a_2) = \mu_{a_2}(f(l, a_1))f(l, a_2)$ for all $l \in L$ and $a_1, a_2 \in A$.*

Proposition 2.20 ([6, Proposition 3.7]). *Let $\mathcal{E}$ be an abelian extension of relative Rota–Baxter groups. Then the following assertions hold:*

1. *The map $\nu : B \to \mathrm{Aut}(K)$ defined by*

$$\nu_b(k) = \phi_{s_G(b)}(k) \tag{16}$$

for $b \in B$ and $k \in K$, is a homomorphism of groups.

2. *The $\mu : A \to \mathrm{Aut}(K)$ defined by*

$$\mu_a(k) = s_H(a)^{-1}\, k\, s_H(a) \tag{17}$$

for $a \in A$ and $k \in K$, is an anti-homomorphism of groups.

3. *The map $\sigma : B \to \mathrm{Aut}(L)$ defined by*

$$\sigma_b(l) = s_G(b)^{-1}\, l\, s_G(b) \tag{18}$$

for $b \in B$ and $l \in K$, is an anti-homomorphism of groups.

Further, all the maps are independent of the choice of a section to $\mathcal{E}$.

4. *The map $\tau_1 : A \times A \to K$ given by*

$$\tau_1(a_1, a_2) = s_H(a_1a_2)^{-1}s_H(a_1)s_H(a_2) \tag{19}$$

for $a_1, a_2 \in A$ is a group 2-cocycle with respect to the action μ.

5. *The map $\tau_2 : B \times B \to L$ given by*

$$\tau_2(b_1, b_2) = s_G(b_1b_2)^{-1}s_G(b_1)s_G(b_2) \tag{20}$$

for $b_1, b_2 \in B$ is a group 2-cocycle with respect to the action σ.

By examining the relationships between ν, μ, σ, f and by their properties outlined in Lemma 2.19 and Proposition 2.20, we are led to the following definition of a module over a relative Rota–Baxter group [6, Definition 3.12].

Definition 2.21. A module over a relative Rota–Baxter group (A, B, β, T) is a trivial relative Rota–Baxter group (K, L, α, S) such that there exists a quadruple (ν, μ, σ, f) of maps (called an action) satisfying the following conditions:

1. The group K is a left B-module and a right A-module with respect to the actions $\nu : B \to \mathrm{Aut}(K)$ and $\mu : A \to \mathrm{Aut}(K)$, respectively.

2. The group L is a right B-module with respect to the action $\sigma : B \to \mathrm{Aut}(L)$.

3. The map $f : L \times A \to K$ has the property that $f(-, a) : L \to K$ is a homomorphism for all $a \in A$ and $f(l, -) : A \to K$ is a derivation with respect to the action μ for all $l \in L$.

4. $S\big(\nu_{T(a)}^{-1}(\mu_a(k))\,\nu_{T(a)}^{-1}(f(S(k),a))\big) = \sigma_{T(a)}(S(k))$ for all $a \in A$ and $k \in K$.

5. $\nu_b(\mu_a(k)) = \mu_{\beta_b(a)}(\nu_b(k))$ for all $a \in A$, $b \in B$ and $k \in K$.

Let G be a group and M an abelian group. Let $C^n(G,M)$ denote the group of maps $G^n \to M$ that vanish on degenerate tuples. Similarly, for another group H, let $C(G \times H, M)$ denote the group of maps $G \times H \to M$ that vanish on degenerate tuples.

Let $\mathcal{K} = (K, L, \alpha, S)$ be an $\mathcal{A} = (A, B, \beta, T)$-module with respect to the action (ν, μ, σ, f). We set

$$\begin{aligned}
\mathrm{C}^1_{RRB}(\mathcal{A},\mathcal{K}) &:= C^1(A,K) \oplus C^1(B,L) \text{ and}\\
\mathrm{C}^2_{RRB}(\mathcal{A},\mathcal{K}) &:= C^2(A,K) \oplus C^2(B,L) \oplus C(A \times B, K) \oplus C(A,L).
\end{aligned}$$

Let $\mathrm{Z}^2_{RRB}(\mathcal{A},\mathcal{K})$ be the subgroup of $\mathrm{C}^2_{RRB}(\mathcal{A},\mathcal{K})$ consisting of the elements $(\tau_1, \tau_2, \rho, \chi)$ that satisfy the conditions

$$\begin{aligned}
\tau_1(a_2,a_3)\tau_1(a_1,a_2a_3) &= \tau_1(a_1a_2,a_3)\mu_{a_3}(\tau_1(a_1,a_2)), && (21)\\
\tau_2(b_2,b_3)\tau_2(b_1,b_2b_3) &= \tau_2(b_1b_2,b_3)\sigma_{b_3}(\tau_2(b_1,b_2)), && (22)\\
\rho(\beta_{b_2}(a_1),b_1)\,\nu_{b_1}(\rho(a_1,b_2)) &= \rho(a_1,b_1b_2)\,\nu_{b_1b_2}(f(\tau_2(b_1,b_2),a_1)), && (23)\\
\rho(a_1a_2,b_1)\,\nu_{b_1}(\tau_1(a_1,a_2)) &= \mu_{\beta_{b_1}(a_2)}(\rho(a_1,b_1))\,\rho(a_2,b_1)\,\tau_1(\beta_{b_1}(a_1),\beta_{b_1}(a_2)), && (24)\\
\tau_2(T(a_1),T(a_2))\delta^1_\sigma(\chi)(a_1,a_2) &= S\big(\nu^{-1}_{T(a_1\circ_T a_2)}\big(\rho(a_2,T(a_1))\,\tau_1(a_1,\beta_{T(a_1)}(a_2))\\
&\qquad \nu_{T(a_1)}(f(\chi(a_1),a_2))\big)\big) && (25)
\end{aligned}$$

for all $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$, where

$$\delta^1_\sigma(\chi)(a_1,a_2) = \chi(a_2)\chi(a_1 \circ_T a_2)^{-1}\sigma_{T(a_2)}(\chi(a_1)).$$

Next, let $\mathrm{B}^2_{RRB}(\mathcal{A},\mathcal{K})$ be the subgroup of $\mathrm{Z}^2_{RRB}(\mathcal{A},\mathcal{K})$ formed by elements $(\tau_1,\tau_2,\rho,\chi)$ for which there exist $\kappa_1, \kappa_2 \in \mathrm{C}^1_{RRB}(\mathcal{A},\mathcal{K})$ satisfying the conditions

$$\begin{aligned}
\tau_1(a_1,a_2) &= \kappa_1(a_1a_2)^{-1}\kappa_1(a_2)\mu_{a_2}(\kappa_1(a_1)),\\
\tau_2(b_1,b_2) &= \kappa_2(b_1b_2)^{-1}\kappa_2(b_2)\sigma_{b_2}(\kappa_2(b_1)),\\
\rho(a_1,a_2) &= \nu_b\big(f(\kappa_2(b_1),a_1)\kappa_1(a_1)\big)\big(\kappa_1(\beta_{b_1}(a_1))\big)^{-1},\\
\chi(a_1) &= S\big(\nu^{-1}_{T(a_1)}(\kappa_1(a_1))\big)\big(\kappa_2(T(a_1))\big)^{-1}
\end{aligned}$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

We define the second cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{K}$ by

$$\mathrm{H}^2_{RRB}(\mathcal{A},\mathcal{K}) = \mathrm{Z}^2_{RRB}(\mathcal{A},\mathcal{K})/\,\mathrm{B}^2_{RRB}(\mathcal{A},\mathcal{K}).$$

Consider a 2-cocycle $(\tau_1,\tau_2,\rho,\chi) \in \ker(\partial^2_{RRB})$. Define $H = A \times_{\tau_1} K$ and $G = B \times_{\tau_2} L$ to be the group extensions of A by K and B by L associated to the group 2-cocycles τ_1 and τ_2, respectively. Further, define $\phi : G \to \mathrm{Aut}(H)$ by

$$\phi_{(b,l)}(a,k) = \big(\beta_b(a), \rho(a,b)\,\nu_b(f(l,a)k)\big) \tag{26}$$

and $R : H \to G$ by

$$R(a,k) = \big(T(a), \chi(a)\, S(\nu^{-1}_{T(a)}(k))\big) \tag{27}$$

for all $a \in A$, $b \in B$, $k \in K$ and $l \in L$. Using the fact that $(\tau_1, \tau_2, \rho, \chi) \in \ker(\partial^2_{RRB})$, it follows that (H, G, ϕ, R) is a relative Rota–Baxter group and is an extension of (A, B, β, T) by (K, L, α, S) denoted by

$$\mathcal{E}(\tau_1, \tau_2, \rho, \chi): \qquad \mathbf{1} \to (K, L, \alpha, S) \xrightarrow{(i_1, i_2)} (H, G, \phi, R) \xrightarrow{(\pi_1, \pi_2)} (A, B, \beta, T) \to \mathbf{1}.$$

Let $\mathrm{Ext}_{(\nu,\mu,\sigma,f)}(\mathcal{A}, \mathcal{K})$ denote the set of equivalence classes of extensions of $\mathcal{A}$ by $\mathcal{K}$ for which the associated action is (ν, μ, σ, f). Then we have the following result [6, Theorem 3.18].

Theorem 2.22. *Let $\mathcal{A} = (A, B, \beta, T)$ be a relative Rota–Baxter group and $\mathcal{K} = (K, L, \alpha, S)$ be a trivial relative Rota–Baxter group, where K and L are abelian groups. Let (ν, μ, σ, f) be the quadruple of actions turning $\mathcal{K}$ into an $\mathcal{A}$-module. Then there is a bijective function $\Gamma : \mathrm{H}^2_{RRB}(\mathcal{A}, \mathcal{K}) \to \mathrm{Ext}_{(\nu,\mu,\sigma,f)}(\mathcal{A}, \mathcal{K})$ given by*

$$\Gamma\big([\tau_1, \tau_2, \rho, \chi]\big) = [\mathcal{E}(\tau_1, \tau_2, \rho, \chi)].$$

3 From relative Rota–Baxter groups to Rota–Baxter groups on semi-direct products

Recall that, the semi-direct product $H \rtimes_\phi G$ of groups G and H under the (left) action $\phi \colon G \to \mathrm{Aut}(H)$ is the set $H \times G$ equipped with the multiplication

$$(h_1, g_1)(h_2, g_2) = \big(h_1 \phi_{g_1}(h_2), g_1 g_2\big)$$

for all $h_1, h_2 \in H$ and $g_1, g_2 \in G$.

The following result shows that each relative Rota–Baxter group defines a Rota–Baxter group on the natural semi-direct product [4, Proposition 3.1].

Proposition 3.1. *Let (H, G, ϕ, R) be a relative Rota–Baxter group. Then, the function $\widetilde{R} \colon H \rtimes_\phi G \to H \rtimes_\phi G$ given by*

$$\widetilde{R}\big((h, g)\big) = \big(1, g^{-1} R(h)\big)$$

for all $h \in H$ and $g \in G$, is a Rota–Baxter operator on the semi-direct product $H \rtimes_\phi G$.

We observe that the preceding construction is, in fact, functorial.

Proposition 3.2. *Let $\mathcal{RRB}$ and $\mathcal{RB}$ denote the categories of relative Rota–Baxter groups and Rota–Baxter groups, respectively. Then the association $\mathcal{F} : \mathcal{RRB} \to \mathcal{RB}$ given by $\mathcal{F}\big((H, G, \phi, R)\big) = \big(H \rtimes_\phi G, \widetilde{R}\big)$ is a covariant functor.*

Proof. Let $(f_1, f_2) : (H, G, \phi, R) \to (K, L, \alpha, S)$ be a homomorphism of relative Rota–Baxter groups. Define $\widetilde{f} : H \rtimes_\phi G \to K \rtimes_\alpha L$ by $\widetilde{f}(h, g) = (f_1(h), f_2(g))$ for all $h \in H$ and $g \in G$. We claim that $\widetilde{f} : (H \rtimes_\phi G, \widetilde{R}) \to (K \rtimes_\alpha L, \widetilde{S})$ is a homomorphism of Rota–Baxter groups. Indeed, for $g, g' \in G$ and $h, h' \in H$, we have

$$\begin{aligned}
\widetilde{f}\big((h,g)(h',g')\big) &= \widetilde{f}\big((h\phi_g(h'), gg')\big)\\
&= \big(f_1(h\phi_g(h')), f_2(gg')\big)\\
&= \big(f_1(h)f_1(\phi_g(h')), f_2(gg')\big)\\
&= \big(f_1(h)\alpha_{f_2(g)}(f_1(h')), f_2(gg')\big),\\
&\quad \text{since } (f_1, f_2) \text{ is a homomorphism of relative Rota–Baxter groups}\\
&= \big(f_1(h), f_2(g)\big)\big(f_1(h'), f_2(g')\big)\\
&= \widetilde{f}\big((h,g)\big)\widetilde{f}\big((h',g')\big)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{f}\,\widetilde{R}\big((h,g)\big) &= \widetilde{f}\big((1, g^{-1}R(h))\big)\\
&= \big(1, f_2(g^{-1}R(h)\big)\\
&= \big(1, f_2(g)^{-1}f_2R(h)\big)\\
&= \big(1, f_2(g)^{-1}Sf_1(h)\big),\\
&\quad \text{since } (f_1, f_2) \text{ is a homomorphism of relative Rota–Baxter groups}\\
&= \widetilde{S}\big(f_1(h), f_2(g)\big)\\
&= \widetilde{S}\,\widetilde{f}\big((h,g)\big).
\end{aligned}$$

It is now immediate to see that $\mathcal{F} : \mathcal{RRB} \to \mathcal{RB}$ is a functor. □

In fact, $\mathcal{F}$ maps equivalent extensions to equivalent extensions.

Corollary 3.3. *Let $\mathcal{A} = (A, B, \beta, T)$ be a relative Rota–Baxter group, $\mathcal{K} = (K, L, \alpha, S)$ a trivial relative Rota–Baxter group such that K and L are abelian groups, and*

$$\mathcal{E}: \quad \mathbf{1} \longrightarrow (K, L, \alpha, S) \xrightarrow{(i_1, i_2)} (H, G, \phi, R) \xrightarrow{(\pi_1, \pi_2)} (A, B, \beta, T) \longrightarrow \mathbf{1}$$

an extension of relative Rota–Baxter groups with associated action (ν, μ, σ, f). Then

$$\widetilde{\mathcal{E}}: \quad \mathit{1} \longrightarrow (K \rtimes_\alpha L, \widetilde{S}) \xrightarrow{\widetilde{i}} (H \rtimes_\phi G, \widetilde{R}) \xrightarrow{\widetilde{\pi}} (A \rtimes_\beta B, \widetilde{T}) \longrightarrow \mathit{1}$$

is an extension of Rota–Baxter groups with associated action γ given by

$$\gamma_{(a,b)}(k, l) = \big(\nu_b^{-1}(\mu_a(k)f(l, a)),\ \sigma_b(l)\big) \tag{28}$$

for all $a \in A$, $b \in B$, $k \in K$ and $l \in L$. Further, there is a map

$$\Pi : \operatorname{Ext}_{(\nu,\mu,\sigma,f)}(\mathcal{A}, \mathcal{K}) \to \operatorname{Ext}_\gamma(A \rtimes_\beta B, K \times L) \quad \text{given by} \quad \Pi\big([\mathcal{E}]\big) = [\widetilde{\mathcal{E}}].$$

Proof. Let (s_H, s_G) be a set-theoretic section to $\mathcal{E}$ inducing the action (ν, μ, σ, f). Then the map $s((a,b)) = (s_H(a), s_G(b))$ is a set-theoretic section to $\widetilde{\mathcal{E}}$. Let γ be the action associated to the extension $\mathcal{E}$. For $a \in A$, $b \in B$, $k \in K$ and $l \in L$, we see that

$$\begin{aligned}
\gamma_{(a,b)}(k,l) &= s(a,b)^{-1}(k,l)s(a,b)\\
&= \big(s_H(a),\ s_G(b)\big)^{-1}(k,\ l)\big(s_H(a),\ s_G(b)\big)\\
&= \big((\phi_{s_G(b)^{-1}}(s_H(a)))^{-1},\ s_G(b)^{-1}\big)\big(k\,\phi_l(s_H(a)),\ l s_G(b)\big)\\
&= \big((\phi_{s_G(b)^{-1}}(s_H(a)))^{-1}\phi_{s_G(b)^{-1}}(k\,\phi_l(s_H(a))),\ s_G(b)^{-1} l s_G(b)\big)\\
&= \big(\phi_{s_G(b)^{-1}}\big(s_H(a)^{-1}k\,\phi_l(s_H(a))\big),\ \sigma_b(l)\big), \text{ by (18)}\\
&= \big(\phi_{s_G(b)^{-1}}\big(s_H(a)^{-1}k\,s_H(a)s_H(a)^{-1}\phi_l(s_H(a))\big),\ \sigma_b(l)\big)\\
&= \big(\phi_{s_G(b)^{-1}}\big(\mu_a(k)f(l,a)\big),\ \sigma_b(l)\big), \text{ by Lemma (2.19) and (17)}\\
&= \big(\nu_b^{-1}(\mu_a(k)f(l,a)),\ \sigma_b(l)\big), \text{ by (16).}
\end{aligned}$$

Hence, the associated action γ of the induced extension $\widetilde{\mathcal{E}}$ is completely determined by the action (ν, μ, σ, f) of the extension $\mathcal{E}$. A direct check shows that equivalent extensions of relative Rota–Baxter groups map to equivalent extensions of Rota–Baxter groups. By [6, Proposition 3.17], equivalent extensions of relative Rota–Baxter groups induce identical associated actions, and the result follows. □

Let $\mathcal{A} = (A, B, \beta, T)$ and $\mathcal{K} = (K, L, \alpha, S)$ be as in Corollary 3.3. Define the map $\tau\colon (A \rtimes_\beta B) \times (A \rtimes_\beta B) \to K \times L$ by

$$\tau\big((a_1,b_1),(a_2,b_2)\big) = s\big((a_1,b_1)(a_2,b_2)\big)^{-1} s\big((a_1,b_1)\big) s\big((a_2,b_2)\big)$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then, we have

$$\begin{aligned}
&\tau\big((a_1,b_1),(a_2,b_2)\big)\\
&\quad= s\big((a_1\beta_{b_1}(a_2),\ b_1b_2)\big)^{-1}\big(s_H(a_1),\ s_G(b_1)\big)\big(s_H(a_2),\ s_G(b_2)\big)\\
&\quad= (s_H(a_1\beta_{b_1}(a_2)),\ s_G(b_1b_2))^{-1}\ \big(s_H(a_1)\phi_{s_G(b_1)}(s_H(a_2)),\ s_G(b_1)s_G(b_2)\big)\\
&\quad= \big(\phi_{s_G(b_1b_2)^{-1}}(s_H(a_1\beta_{b_1}(a_1)))^{-1},\ s_G(b_1b_2)^{-1}\big)\ \big(s_H(a_1)\phi_{s_G(b_1)}(s_H(a_2)),\ s_G(b_1)s_G(b_2)\big)\\
&\quad= \big(\phi_{s_G(b_1b_2)^{-1}}(s_H(a_1\beta_{b_1}(a_1)))^{-1}\phi_{s_G(b_1b_2)^{-1}}(s_H(a_1)\phi_{s_G(b_1)}(s_H(a_2))),\\
&\qquad\quad s_G(b_1b_2)^{-1}s_G(b_1)s_G(b_2)\big)\\
&\quad= \big(\phi_{s_G(b_1b_2)^{-1}}\big((s_H(a_1\beta_{b_1}(a_1)))^{-1}s_H(a_1)\phi_{s_G(b_1)}(s_H(a_2))\big),\ \tau_2(b_1,b_2)\big), \text{using (20)}\\
&\quad= \big(\phi_{s_G(b_1b_2)^{-1}}\big(\tau_1(a_1,\beta_{b_1}(a_2))s_H(\beta_{b_1}(a_2))^{-1}\phi_{s_G(b_1)}(s_H(a_2))\big),\ \tau_2(b_1,b_2)\big), \text{ using (19)}\\
&\quad= \big(\phi_{s_G(b_1b_2)^{-1}}\big(\tau_1(a_1,\beta_{b_1}(a_2))\rho(a_2,b_1)\big),\ \tau_2(b_1,b_2)\big), \text{using Proposition (2.18)(1)}\\
&\quad= \big(\nu_{b_1b_2}^{-1}\big(\tau_1(a_1,\beta_{b_1}(a_2))\rho(a_2,b_1)\big),\ \tau_2(b_1,b_2)\big)\,.
\end{aligned} \tag{29}$$

Now, define the map $r\colon A \rtimes_\beta B \to K \times L$ by

$$r(a,b) = s\big(\widetilde{T}(a,b)\big)^{-1}\,\widetilde{R}\big(s(a,b)\big)$$

for all $a \in A$ and $b \in B$. Then, we have

$$\begin{aligned}
r(a,b) &= s\big(1, b^{-1}T(a)\big)^{-1}\, \widetilde{R}\big(s_H(a), s_G(b)\big)\\
&= \big(1, s_G(b^{-1}T(a))\big)^{-1}\big(1, s_G(b)^{-1}R(s_H(a))\big)\\
&= \big(1, s_G(b^{-1}T(a))^{-1}\big)\big(1, s_G(b)^{-1}R(s_H(a))\big)\\
&= \big(1, s_G(b^{-1}T(a))^{-1}s_G(b)^{-1}R(s_H(a))\big)\\
&= \big(1, \tau_2(b^{-1}, T(a))s_G(T(a))^{-1}s_G(b^{-1})^{-1}s_G(b)^{-1}R(s_H(a))\big), \text{using } (20)\\
&= \big(1, \tau_2(b^{-1}, T(a))s_G(T(a))^{-1}\tau_2(b, b^{-1})^{-1}s_G(T(a))\chi(a)\big),\\
&\quad\ \text{using Proposition } (2.18)(2) \text{ and } (20)\\
&= \big(1, \tau_2(b^{-1}, T(a))\sigma_{T(a)}(\tau_2(b, b^{-1})^{-1})\chi(a)\big), \text{using } (18)\\
&= \big(1, \tau_2(b, b^{-1}T(a))^{-1}\chi(a)\big), \text{using } (22).
\end{aligned} \tag{30}$$

Next, we explore the relationship between $\mathrm{H}^2_{RRB}(\mathcal{A}, \mathcal{K})$ and $\mathrm{H}^2_{RB}(A \rtimes_\beta B, K \times L)$ (see Subsections 2.2 and 2.3 for the constructions). Let $\Gamma : \mathrm{H}^2_{RRB}(\mathcal{A}, \mathcal{K}) \to \mathrm{Ext}_{(\nu,\mu,\sigma,f)}(\mathcal{A}, \mathcal{K})$ and $\Lambda : \mathrm{Ext}_\gamma(A \rtimes_\beta B, K \times L) \to \mathrm{H}^2_{RB}(A \rtimes_\beta B, K \times L)$ be the bijections defined in Theorems 2.22 and 2.12, respectively. Let $\Pi : \mathrm{Ext}_{(\nu,\mu,\sigma,f)}(\mathcal{A}, \mathcal{K}) \to \mathrm{Ext}_\gamma(A \rtimes_\beta B, K \times L)$ be the map defined in Corollary 3.3. Then we have the following result.

Proposition 3.4. *Let $\mathcal{A} = (A, B, \beta, T)$ be a relative Rota–Baxter group and assume that $\mathcal{K} = (K, L, \alpha, S)$ is a module over $\mathcal{A}$ with respect to the action (ν, μ, σ, f). Then the map*

$$\Omega_{RB} := \Lambda\Pi\Gamma : \mathrm{H}^2_{RRB}(\mathcal{A}, \mathcal{K}) \to \mathrm{H}^2_{RB}(A \rtimes_\beta B, K \times L)$$

is a homomorphism of groups.

Proof. The map Ω_{RB} is given by $\Omega_{RB}\big([\tau_1, \tau_2, \rho, \chi]\big) = [\tau^{(\tau_1,\tau_2,\rho,\chi)},\ r^{(\tau_1,\tau_2,\rho,\chi)}]$, where

$$\begin{aligned}
\tau^{(\tau_1,\tau_2,\rho,\chi)}\big((a_1, b_1), (a_2, b_2)\big) &= \big(\nu^{-1}_{b_1b_2}\big(\tau_1(a_1, \beta_{b_1}(a_2))\rho(a_2, b_1)\big), \tau_2(b_1, b_2)\big),\\
r^{(\tau_1,\tau_2,\rho,\chi)}(a, b) &= \big(1, \tau_2(b, b^{-1}T(a))^{-1}\chi(a)\big)
\end{aligned}$$

for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Let $[\tau_1, \tau_2, \rho, \chi]$ and $[\tau_1', \tau_2', \rho', \chi']$ be elements of $\mathrm{H}^2_{RRB}(\mathcal{A}, \mathcal{K})$. Then, we have

$$\begin{aligned}
\Omega_{RB}\big([\tau_1, \tau_2, \rho, \chi]\,[\tau_1', \tau_2', \rho', \chi']\big) &= \Omega_{RB}\big([\tau_1\tau_1', \tau_2\tau_2', \rho\rho', \chi\chi']\big)\\
&= \big[\tau^{(\tau_1\tau_1', \tau_2\tau_2', \rho\rho', \chi\chi')},\ r^{(\tau_1\tau_1', \tau_2\tau_2', \rho\rho', \chi\chi')}\big].
\end{aligned}$$

It is easy to see that

$$\tau^{(\tau_1\tau_1', \tau_2\tau_2', \rho\rho', \chi\chi')} = \tau^{(\tau_1,\tau_2,\rho,\chi)}\,\tau^{(\tau_1',\tau_2',\rho',\chi')} \quad \text{and} \quad r^{(\tau_1\tau_1', \tau_2\tau_2', \rho\rho', \chi\chi')} = r^{(\tau_1,\tau_2,\rho,\chi)}\,r^{(\tau_1',\tau_2',\rho',\chi')}.$$

Hence, we obtain

$$\begin{aligned}
\Omega_{RB}\big([\tau_1, \tau_2, \rho, \chi][\tau_1', \tau_2', \rho', \chi']\big) &= \big[\tau^{(\tau_1,\tau_2,\rho,\chi)}, r^{(\tau_1,\tau_2,\rho,\chi)}\big]\,\big[\tau^{(\tau_1',\tau_2',\rho',\chi')}, r^{(\tau_1',\tau_2',\rho',\chi')}\big]\\
&= \Omega_{RB}\big([\tau_1, \tau_2, \rho, \chi]\big)\,\Omega_{RB}\big([\tau_1', \tau_2', \rho', \chi']\big),
\end{aligned}$$

which is desired. □

4 From skew left braces to skew left braces on semi-direct products

It is known from [23, Theorem 3.10] that if $(H,\cdot,\circ)$ is a skew left brace, then the quadruple $(H^{(\cdot)}, H^{(\circ)}, \lambda^H, \mathrm{id}_H)$ is a relative Rota–Baxter group, where

$$\lambda^H : H^{(\circ)} \to \mathrm{Aut}(H^{(\cdot)}) \quad \text{is defined by} \quad \lambda^H_a(b) = a^{-1}\cdot(a\circ b)$$

for all $a,b\in H$. Furthermore, the skew left brace induced by the relative Rota–Baxter group $(H^{(\cdot)}, H^{(\circ)}, \lambda^H, \mathrm{id}_H)$ is the same as $(H,\cdot,\circ)$. Also, if $(H^{(\cdot)}, H^{(\circ)}, \lambda^H, \mathrm{id}_H)$ is a relative Rota–Baxter group, then $(H,\cdot,\circ)$ is a skew left brace with λ^H as its associated action.

Let $(H,\cdot,\circ)$ be a skew left brace and let

$$\widetilde{H} = H^{(\cdot)} \rtimes_{\lambda^H} H^{(\circ)}$$

denote the associated semi-direct product. In view of Proposition 3.1 and Remark 2.14, there is a skew left brace structure on $\widetilde{H}$ induced by the Rota–Baxter group $(\widetilde{H},\widetilde{R})$, where $\widetilde{R}:\widetilde{H}\to\widetilde{H}$ is given by

$$\widetilde{R}(h,g) = (1, g^{\dagger}\circ h) \tag{31}$$

for all $h,g\in H$. The additive group operation $\bullet$ is given by

$$(h_1,g_1)\bullet(h_2,g_2) = \big(h_1\cdot\lambda^H_{g_1}(h_2),\, g_1\circ g_2\big),$$

and the multiplicative group operation $\odot$ is given by

$$(h_1,g_1)\odot(h_2,g_2) = \big(h_1\circ h_2,\, h_1\circ g_2\circ h_1^{\dagger}\circ g_1\big)$$

for all $g_1,g_2,h_1,h_2\in H$.

Definition 4.1. Given a skew left brace $(H,\cdot,\circ)$, the skew left brace $(\widetilde{H},\bullet,\odot)$ is called the *square* of $(H,\cdot,\circ)$.

Remark 4.2. By definition, the square of a skew left brace always arises from a Rota–Baxter group. Further, the associated action for $(\widetilde{H},\bullet,\odot)$ is given by

$$\lambda^{\widetilde{H}}_{(h_1,g_1)}(h_2,g_2) = \widetilde{R}(h_1,g_1)\bullet(h_2,g_2)\bullet\big(\widetilde{R}(h_1,g_1)\big)^{-1} = (1,g_1^{\dagger}\circ h_1)\bullet(h_2,g_2)\bullet(1,g_1^{\dagger}\circ h_1)^{-1}.$$

Example 4.3. Let $(H,\cdot,\cdot)$ be a trivial skew left brace. Then, we have

$$\begin{aligned}(h_1,g_1)\bullet(h_2,g_2) &= \big(h_1\cdot h_2,\, g_1\cdot g_2\big) \text{ and}\\ (h_1,g_1)\odot(h_2,g_2) &= \big(h_1\cdot h_2,\, h_1\cdot g_2\cdot h_1^{-1}\cdot g_1\big).\end{aligned}$$

Thus, the square of a trivial skew left brace need not be a trivial skew left brace. However, the square of a trivial left brace is always trivial.

Recall the following definition from [25, Corollary 3.38].

Definition 4.4. Let $(H, \cdot, \circ)$ be a skew left brace such that

$$\lambda_h^H \lambda_g^H = \lambda^H_{\lambda_h^H(g)} \lambda_h^H$$

for all $g, h \in H$. Then $H \times H$ admits a skew left brace structure, where

$$(h, g) \cdot (h', g') = (h \cdot h',\ g \cdot g') \quad \text{and} \quad (h, g) \circ (h', g') = \big(h \circ h',\ g \circ \lambda_h^H(g')\big).$$

This skew left brace is called the double of $(H, \cdot, \circ)$ and is denoted by $(D(H), \cdot, \circ)$.

Remark 4.5. It follows from the definition that the double of a trivial skew left brace is always a trivial skew left brace. If $(H, \cdot, \circ)$ is a skew left brace, then the associated action for the double $(D(H), \cdot, \circ)$ is given by

$$\lambda^{D(H)}_{(h,g)}(h', g') = \big(\lambda_h^H(h'),\ \lambda^H_{g \circ h}(g')\big).$$

Thus, if λ_h^H is not an inner automorphism of $H^{(\cdot)}$ for some h, then $\lambda^{D(H)}_{(h,g)}$ is not an inner automorphism of $D(H)^{(\cdot)}$ for all (h, g). By (11), the necessary condition for a skew left brace to be induced from a Rota–Baxter operator is that its associated action must take values in the inner automorphism group of its additive group. Thus, if λ_h^H is not an inner automorphism of $H^{(\cdot)}$ for some h, then by Remark 4.2, the double $(D(H), \cdot, \circ)$ and the square $(\widetilde{H}, \bullet, \odot)$ are distinct.

Remark 4.6. By a similar argument, the square of a skew left brace is seen to differ from the double semi-direct product of skew left braces [11, Corollary 21] and the *twofold semi-direct products of skew left braces* [13, Proposition 2.1], where the latter relates to *the semi-direct product of digroups* [16].

Example 4.7. Consider the skew left brace $(\mathbb{Z}, +, \circ)$, where $n \circ m = n + (-1)^n m$. Note that $\lambda_n^{\mathbb{Z}}(m) = (-1)^n m$, and hence $\lambda_n^{\mathbb{Z}}$ is not an inner automorphism of $\mathbb{Z}^{(+)}$ for each $n \in \mathbb{Z}$. For the square $(\widetilde{\mathbb{Z}}, \bullet, \odot)$, we have

$$\begin{aligned}
(n_1, m_1) \bullet (n_2, m_2) &= \big(n_1 + (-1)^{m_1} n_2,\ m_1 + (-1)^{m_1} m_2\big) \text{ and} \\
(n_1, m_1) \odot (n_2, m_2) &= \big(n_1 + (-1)^{n_1} n_2, n_1 + (-1)^{n_1} m_2 + (-1)^{m_2}(m_1 - n_1)\big).
\end{aligned}$$

On the other hand, for the double $(D(\mathbb{Z}), \cdot, \circ)$, we have

$$\begin{aligned}
(n_1, m_1) \cdot (n_2, m_2) &= (n_1 + n_2, m_1 + m_2) \text{ and} \\
(n_1, m_1) \circ (n_2, m_2) &= \big(n_1 + (-1)^{n_1} n_2, m_1 + (-1)^{n_1 + m_1} m_2\big).
\end{aligned}$$

Thus, the square and the double of the skew brace $(\mathbb{Z}, +, \circ)$ are not isomorphic to each other.

We prove that the association $(H, \cdot, \circ) \mapsto (\widetilde{H}, \bullet, \odot)$ is functorial.

Proposition 4.8. *Let $\mathcal{SB}$ denote the category of skew left braces. Then the association $\mathcal{O} : \mathcal{SB} \to \mathcal{SB}$ given by $\mathcal{O}((H, \cdot, \circ)) = (\widetilde{H}, \bullet, \odot)$ is a covariant functor.*

Proof. Let $f \colon (H, \cdot, \circ) \to (G, \cdot, \circ)$ be a morphism of skew left braces. Define

$$\tilde{f} : (\tilde{H}, \bullet, \odot) \to (\tilde{G}, \bullet, \odot) \quad \text{by} \quad \widetilde{f}\big((h, g)\big) = \big(f(h), f(g)\big)$$

for all $g, h \in H$. We claim that $\widetilde{f}$ is a morphism of skew left braces. Indeed, we have

$$\begin{aligned}
\widetilde{f}\big((h,g) \bullet (h',g')\big) &= \widetilde{f}\big((h \cdot \lambda^H_g(h'),\ g \circ g')\big) \\
&= \big(f(h \cdot \lambda^H_g(h')),\ f(g \circ g')\big) \\
&= \big(f(h) \cdot f(\lambda^H_g(h')),\ f(g) \circ f(g')\big), \\
& \text{since } f \text{ is a homomorphism of skew left braces} \\
&= \big(f(h)\, \lambda^G_{f(g)}(f(h')),\ f(g) \circ f(g')\big), \\
& \text{since } f \text{ is a homomorphism of skew left braces} \\
&= \big(f(h), f(g)\big) \bullet \big(f(h'), f(g')\big) \\
&= \widetilde{f}\big((h,g)\big) \bullet \widetilde{f}\big((h',g')\big)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{f}\big((h,g) \odot (h',g')\big) &= \widetilde{f}\big((h \circ h',\ h \circ g' \circ h^\dagger \circ g)\big) \\
&= \big(f(h \circ h'),\ f(h \circ g' \circ h^\dagger \circ g)\big) \\
&= \big(f(h), f(g)\big) \odot \big(f(h'), f(g')\big) \\
&= \widetilde{f}\big((h,g)\big) \odot f\big((h',g')\big),
\end{aligned}$$

for $g, g', h, h' \in H$. It is now immediate to see that $\mathcal{O} : \mathcal{SB} \to \mathcal{SB}$ is a functor. ☐

In fact, $\mathcal{O}$ maps equivalent extensions to equivalent extensions.

Corollary 4.9. *Let $(H, \cdot, \circ)$ be a skew left brace and $(I, \cdot)$ an abelian group viewed as a trivial brace. Let*

$$\mathcal{E} : \quad \mathbf{1} \longrightarrow (I, \cdot) \xrightarrow{i} (E, \cdot, \circ) \xrightarrow{\pi} (H, \cdot, \circ) \longrightarrow \mathbf{1}$$

be an extension of skew left braces with associated action (ξ, ζ, ϵ) and

$$\widetilde{\mathcal{E}} : \quad \mathbf{1} \longrightarrow (\widetilde{I}, \cdot) \xrightarrow{\widetilde{i}} (\widetilde{E}, \bullet, \odot) \xrightarrow{\widetilde{\pi}} (\widetilde{H}, \bullet, \odot) \longrightarrow \mathbf{1}$$

the induced extension of skew left braces with associated action $(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\epsilon})$. Then there is a map $\Theta : \mathrm{Ext}_{(\xi,\zeta,\epsilon)}(H, I) \to \mathrm{Ext}_{(\widetilde{\xi},\widetilde{\zeta},\widetilde{\epsilon})}(\widetilde{H}, \widetilde{I})$ given by $\Theta\big([\mathcal{E}]\big) = [\widetilde{\mathcal{E}}]$.

Proof. Note that, since I is a trivial brace, $\widetilde{I} = I \times I$ is again a trivial brace. A direct check shows that if

$$\mathcal{E}: \quad \mathbf{1} \longrightarrow (I, \cdot) \xrightarrow{i} (E, \cdot, \circ) \xrightarrow{\pi} (H, \cdot, \circ) \longrightarrow \mathbf{1}$$

is an abelian extension of skew left braces, then the induced sequence

$$\widetilde{\mathcal{E}}: \quad \mathbf{1} \longrightarrow (\widetilde{I}, \cdot) \xrightarrow{\widetilde{i}} (\widetilde{E}, \bullet, \odot) \xrightarrow{\widetilde{\pi}} (\widetilde{H}, \bullet, \odot) \longrightarrow \mathbf{1}$$

of skew left braces is also exact. Let s be a set-theoretic section to $\mathcal{E}$. Then, we have that $\widetilde{s}(h, g) = (s(h), s(g))$ is a set-theoretic section to $\widetilde{\mathcal{E}}$. Let $(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\epsilon})$ be the associated action of $\widetilde{\mathcal{E}}$. For $a, b \in I$ and $h, g \in H$, we see that

$$\begin{aligned}
\widetilde{\xi}_{(h,g)}(a, b) &= \lambda^{\widetilde{E}}_{(s(h),s(g))}(a, b) \\
&= \big(1, s(g)^{\dagger} \circ s(h)\big) \bullet (a, b) \bullet \big(1, s(h)^{\dagger} \circ s(g)\big), \text{by definition of } \lambda^{\widetilde{E}}_{(s(h),s(g))} \\
&= \big(1, s(g)^{\dagger} \circ s(h)\big) \bullet \big(a, b \circ s(h)^{\dagger} \circ s(g)\big) \\
&= \big(\lambda^{E}_{s(g)^{\dagger}}(\lambda^{E}_{s(h)}(a)),\ s(g)^{\dagger} \circ s(h) \circ b \circ s(h)^{\dagger} \circ s(g)\big) \\
&= \big(\xi_g^{-1}(\xi_h(a)),\ \epsilon_g(s(h) \circ b \circ s(h)^{\dagger})\big), \text{by (1) and (3)} \\
&= \big(\xi_g^{-1}(\xi_h(a)),\ \epsilon_g(\epsilon_h^{-1}(b))\big), \text{using (3)} \\
&= \big(\xi_{(g^{\dagger} \circ h)}(a),\ \epsilon_{(h^{\dagger} \circ g)}(b)\big), \qquad (32)
\end{aligned}$$

where the last equality follows from the fact that ξ is a homomorphism and ϵ is an anti-homomorphism.

To compute $\widetilde{\zeta}$, we see that

$$\begin{aligned}
\widetilde{\zeta}_{(h,g)}(a, b) &= (\widetilde{s}(h, g))^{-1} \bullet (a, b) \bullet \widetilde{s}(h, g) \\
&= (s(h), s(g))^{-1} \bullet (a, b) \bullet (s(h), s(g)) \\
&= \big(\lambda^{E}_{s(g)^{\dagger}}(s(h))^{-1}, s(g)^{\dagger}\big) \bullet \big(a \cdot \lambda^{E}_{b}(s(h)), b \circ s(g)\big) \\
&= \big(\lambda^{E}_{s(g)^{\dagger}}(s(h)^{-1} \cdot a\ \cdot \lambda^{E}_{b}(s(h))),\ s(g)^{\dagger} \circ b \circ s(g)\big) \\
&= \big(\lambda^{E}_{s(g)^{\dagger}}(s(h)^{-1} \cdot a\ \cdot s(h) \cdot s(h)^{-1} \cdot \lambda^{E}_{b}(s(h))),\ \epsilon_g(b)\big), \text{ by (3).} \qquad (33)
\end{aligned}$$

Note that,

$$\begin{aligned}
s(h)^{-1} \cdot \lambda^{E}_{b}(s(h)) &= s(h)^{-1} \cdot \big(b^{-1} \cdot (b \circ s(h))\big) \\
&= s(h)^{-1} \cdot b^{-1} \cdot s(h) \cdot \lambda^{E}_{s(h)}\big(s(h)^{\dagger} \circ b \circ s(h)\big) \\
&= \zeta_h(b^{-1})\, \xi_h\big(\epsilon_h(b)\big), \text{using (1), (2) and (3).}
\end{aligned}$$

Using the above equation in (33), we get

$$\widetilde{\zeta}_{(h,g)}(a, b) = \big(\lambda^{E}_{s(g)^{\dagger}}(\zeta_h(a)\zeta_h(b^{-1})\xi_h(\epsilon_h(b))),\ \epsilon_g(b)\big), \text{using (2)}$$

$$= \quad \big(\xi_g^{-1}\big(\zeta_h(a)\zeta_h(b^{-1})\xi_h(\epsilon_h(b))\big),\ \epsilon_g(b)\big), \text{using (1)}. \tag{34}$$

Similarly, to determine $\widetilde{\epsilon}$, we have

$$\begin{aligned}
\widetilde{\epsilon}_{(h,g)}(a,b) &= (\widetilde{s}(h,g))^{\dagger} \odot (a,b) \odot \widetilde{s}(h,g)\\
&= (s(h),s(g))^{\dagger} \odot (a,b) \odot (s(h),s(g))\\
&= \big(s(h)^{\dagger}, s(h)^{\dagger} \circ s(g)^{\dagger} \circ s(h)\big) \odot \big(a \circ s(h), a \circ s(g) \circ a^{\dagger} \circ b\big)\\
&= \big(s(h)^{\dagger} \circ a \circ s(h),\ s(h)^{\dagger} \circ a \circ s(g) \circ a^{\dagger} \circ b \circ s(g)^{\dagger} \circ s(h)\big)\\
&= \big(\epsilon_h(a), s(h)^{\dagger} \circ a \circ \epsilon_g^{-1}(a^{\dagger} \circ b) \circ s(h)\big), \text{using (3)}\\
&= \big(\epsilon_h(a),\ \epsilon_h(a \circ \epsilon_g^{-1}(a^{\dagger} \circ b))\big).
\end{aligned} \tag{35}$$

Hence, we conclude that the associated action $(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\epsilon})$ of the induced skew left brace is completely determined by the action (ξ, ζ, ϵ) of the given skew left brace. A direct check shows that equivalent extensions map to equivalent extensions. By [24, Proposition 3.4], equivalent extensions induce identical associated actions, and the result follows. □

Let $(H, \cdot, \circ)$ be a skew left brace and I be an abelian group viewed as a trivial brace. We show that there exists a group homomorphism $\mathrm{H}^2_{SB}(H, I) \to \mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I})$.

Let $\mathcal{E}$ be an extension of H by I and s be a set-theoretic section to $\mathcal{E}$. Let $\widetilde{\mathcal{E}}$ be the induced extension of $\widetilde{H}$ by $\widetilde{I}$ and $\widetilde{s}$ given by $\widetilde{s}(h,g) = (s(h), s(g))$ be the induced set-theoretic section to $\widetilde{\mathcal{E}}$. Define $\tau^{(\tau_1,\tau_2)} \colon \widetilde{H} \times \widetilde{H} \to \widetilde{I}$ by

$$\tau^{(\tau_1,\tau_2)}\big((h_1,g_1),(h_2,g_2)\big) = \widetilde{s}\big((h_1,g_1) \bullet (h_2,g_2)\big)^{-1} \bullet \widetilde{s}(h_1,g_1) \bullet \widetilde{s}(h_2,g_2)$$

for all $g_1, g_2, h_1, h_2 \in H$. Then, we have

$$\begin{aligned}
&\tau^{(\tau_1,\tau_2)}\big((h_1,g_1),(h_2,g_2)\big)\\
&\quad = \big(\widetilde{s}(h_1 \cdot \lambda^H_{g_1}(h_2), g_1 \circ g_2)\big)^{-1} \bullet \big(s(h_1), s(g_1)\big) \bullet \big(s(h_2), s(g_2)\big)\\
&\quad = \big(s(h_1 \cdot \lambda^H_{g_1}(h_2)), s(g_1 \circ g_2)\big)^{-1} \bullet \big(s(h_1) \cdot \lambda^E_{s(g_1)}(s(h_2)), s(g_1) \circ s(g_2)\big)\\
&\quad = \big(\lambda^E_{s(g_1 \circ g_2)^{\dagger}}(s(h_1 \cdot \lambda^H_{g_1}(h_2)))^{-1}, s(g_1 \circ g_2)^{\dagger}\big) \bullet \big(s(h_1) \cdot \lambda^E_{s(g_1)}(s(h_2)), s(g_1) \circ s(g_2)\big)\\
&\quad = \big(\lambda^E_{s(g_1 \circ g_2)^{\dagger}}\big(s(h_1 \cdot \lambda^H_{g_1}(h_2))^{-1} \cdot s(h_1) \cdot \lambda^E_{s(g_1)}(s(h_2))\big),\ s(g_1 \circ g_2)^{\dagger} \circ s(g_1) \circ s(g_2)\big).
\end{aligned} \tag{36}$$

We examine the terms independently. For the first component, we see that

$$\begin{aligned}
&s\big(h_1 \cdot \lambda^H_{g_1}(h_2)\big)^{-1} \cdot s(h_1) \cdot \lambda^E_{s(g_1)}(s(h_2))\\
&\quad = \tau_1(h_1, \lambda^H_{g_1}(h_2)) \cdot s(\lambda^H_{g_1}(h_2))^{-1} \cdot \lambda^E_{s(g_1)}(s(h_1)), \text{using (9)}\\
&\quad = \tau_1(h_1, \lambda^H_{g_1}(h_2)) \cdot (s(g_1^{-1} \cdot (g_1 \circ h_2)))^{-1} \cdot s(g_1)^{-1} \cdot (s(g_1) \circ s(h_2))\\
&\quad = \tau_1(h_1, \lambda^H_{g_1}(h_2)) \cdot \tau_1(g_1^{-1}, g_1 \circ h_2) \cdot s(g_1 \circ h_2)^{-1} \cdot s(g_1^{-1})^{-1} \cdot s(g_1)^{-1} \cdot (s(g_1) \circ s(h_2)),\\
&\qquad \text{using (9)}
\end{aligned}$$

$$\begin{aligned}
&= \tau_1(h_1, \lambda^H_{g_1}(h_2)) \cdot \tau_1(g_1^{-1}, g_1 \circ h_2) \cdot s(g_1 \circ h_2)^{-1} \cdot \tau_1(g_1, g_1^{-1})^{-1} \cdot (s(g_1) \circ s(h_2))\\
&= \tau_1(h_1, \lambda^H_{g_1}(h_2)) \cdot \tau_1(g_1^{-1}, g_1 \circ h_2) \cdot s(g_1 \circ h_2)^{-1}\\
&\quad \cdot \tau_1(g_1, g_1^{-1})^{-1} \cdot s(g_1 \circ h_2) \cdot s(g_1 \circ h_2)^{-1} \cdot (s(g_1) \circ s(h_2))\\
&= \tau_1(h_1, \lambda^H_{g_1}(h_2)) \cdot \tau_1(g_1^{-1}, g_1 \circ h_2) \cdot \zeta_{(g_1 \circ h_2)}\big(\tau_1(g_1, g_1^{-1})^{-1}\big) \cdot \tau_2(g_1, h_2),
\end{aligned} \tag{37}$$

using (9), (10) and (2).

For the second component, we have

$$\begin{aligned}
& s(g_1 \circ g_2)^{\dagger} \circ \big(s(g_1) \circ s(g_2)\big)\\
={}& s(g_1 \circ g_2)^{\dagger} \cdot \lambda^E_{s(g_1 \circ g_2)^{\dagger}}\big(s(g_1) \circ s(g_2)\big)\\
={}& \lambda_{s(g_1 \circ g_2)^{\dagger}}\big(s(g_1 \circ g_2)^{-1}\big) \cdot \lambda^E_{s(g_1 \circ g_2)^{\dagger}}\big(s(g_1) \circ s(g_2)\big), \text{since } a^{\dagger} = \lambda_a^{-1}(a^{-1}),\\
={}& \lambda_{s(g_1 \circ g_2)^{\dagger}}\big(s(g_1 \circ g_2)^{-1} \cdot (s(g_1) \circ s(g_2))\big)\\
={}& \xi^{-1}_{(g_1 \circ g_2)}(\tau_2(g_1, g_2)).
\end{aligned} \tag{38}$$

Using (37) and (38) in (36), we get

$$\begin{aligned}
&\tau^{(\tau_1,\tau_2)}\big((h_1, g_1),\ (h_2, g_2)\big)\\
&\quad = \Big(\xi^{-1}_{(g_1 \circ g_2)}(\tau_1(h_1, \lambda^H_{g_1}(h_2)) \cdot \tau_1(g_1^{-1}, g_1 \circ h_2) \cdot \zeta_{(g_1 \circ h_2)}(\tau_1(g_1, g_1^{-1})^{-1}) \cdot \tau_2(g_1, h_2)),\\
&\qquad\quad \xi^{-1}_{(g_1 \circ g_2)}(\tau_2(g_1, g_2))\Big).
\end{aligned} \tag{39}$$

Similarly, we define $\tau'^{(\tau_1,\tau_2)} \colon \widetilde{H} \times \widetilde{H} \to \widetilde{I}$ by

$$\tau'^{(\tau_1,\tau_2)}\big((h_1, g_1), (h_2, g_2)\big) = \big(\widetilde{s}((h_1, g_1) \odot (h_2, g_2))\big)^{-1} \bullet \big(\widetilde{s}(h_1, g_1) \odot \widetilde{s}(h_2, g_2)\big).$$

Then, we have

$$\begin{aligned}
&\tau'^{(\tau_1,\tau_2)}\big((h_1, g_1), (h_2, g_2)\big)\\
&\quad = \big(\widetilde{s}(h_1 \circ h_2, h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)\big)^{-1} \bullet \big(s(h_1) \circ s(h_2), s(h_1) \circ s(g_2) \circ s(h_1)^{\dagger} \circ s(g_1)\big)\\
&\quad = \big(s(h_1 \circ h_2),\ s(h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)\big)^{-1} \bullet \big(s(h_1) \circ s(h_2), s(h_1) \circ s(g_2) \circ s(h_1)^{\dagger} \circ s(g_1)\big)\\
&\quad = \big(\lambda^E_{s(h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)^{\dagger}}(s(h_1 \circ h_2)^{-1}),\ s(h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)^{\dagger}\big)\\
&\qquad\quad \big(s(h_1) \circ s(h_2),\ s(h_1) \circ s(g_2) \circ s(h_1)^{\dagger} \circ s(g_1)\big)\\
&\quad = \big(\lambda^E_{s(h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)^{\dagger}}(s(h_1 \circ h_2)^{-1} \cdot (s(h_1) \circ s(h_2))),\\
&\qquad\quad s(h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)^{\dagger} \circ s(h_1) \circ s(g_2) \circ s(h_1)^{\dagger} \circ s(g_1)\big)\\
&\quad = \Big(\xi^{-1}_{(h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)}(\tau_2(h_1, h_2)),\ s(h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)^{\dagger} \circ s(h_1) \circ s(g_2) \circ s(h_1)^{\dagger} \circ s(g_1)\Big),\\
&\qquad\quad \text{using (10) and (1).}
\end{aligned} \tag{40}$$

Computing the second component, we get

$$s(h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)^{\dagger} \circ s(h_1) \circ s(g_2) \circ s(h_1)^{\dagger} \circ s(g_1)$$

$$\begin{aligned}
&= \big(s((h_1 \circ g_2) \circ (h_1^{\dagger} \circ g_1))^{\dagger} \circ s(h_1 \circ g_2) \circ s(h_1^{\dagger} \circ g_1)\big) \circ \\
&\quad s(h_1^{\dagger} \circ g_1)^{\dagger} \circ \big(s(h_1 \circ g_2)^{\dagger} \circ s(h_1) \circ s(g_2)\big) \circ s(h_1)^{\dagger} \circ s(g_1) \\
&= \xi^{-1}_{h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1}\big(\tau_2\big(h_1 \circ g_2, h_1^{\dagger} \circ g_1)\big) \circ s(h_1^{\dagger} \circ g_1)^{\dagger} \circ \xi^{-1}_{(h_1 \circ g_2)}\big(\tau_2(h_1, g_2)\big) \circ s(h_1)^{\dagger} \circ s(g_1), \\
&\quad \text{using (38)} \\
&= \xi^{-1}_{h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1}\big(\tau_2(h_1 \circ g_2, h_1^{\dagger} \circ g_1)\big) \circ s(h_1^{\dagger} \circ g_1)^{\dagger} \circ \xi^{-1}_{(h_1 \circ g_2)}\big(\tau_2(h_1, g_2)\big) \circ s(h_1^{\dagger} \circ g_1) \circ \\
&\quad s(h_1^{\dagger} \circ g_1)^{\dagger} \circ s(h_1)^{\dagger} \circ s(g_1) \\
&= \xi^{-1}_{h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1}\big(\tau_2(h_1 \circ g_2, h_1^{\dagger} \circ g_1)\big) \circ \epsilon_{(h_1^{\dagger} \circ g_1)}\big(\xi^{-1}_{(h_1 \circ g_2)}(\tau_2(h_1, g_2))\big) \circ \\
&\quad s(h_1^{\dagger} \circ g_1)^{\dagger} \circ s(h_1^{\dagger}) \circ \tau_2(h_1, h_1^{\dagger})^{\dagger} \circ s(g_1), \text{using (3)} \\
&= \xi^{-1}_{h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1}\big(\tau_2(h_1 \circ g_2, h_1^{\dagger} \circ g_1)\big) \circ \epsilon_{(h_1^{\dagger} \circ g_1)}\big(\xi^{-1}_{(h_1 \circ g_2)}(\tau_2(h_1, g_2))\big) \circ \\
&\quad s(h_1^{\dagger} \circ g_1)^{\dagger} \circ s(h_1^{\dagger}) \circ s(g_1) \circ s(g_1)^{\dagger} \circ \tau_2(h_1, h_1^{\dagger})^{\dagger} \circ s(g_1) \\
&= \xi^{-1}_{h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1}\big(\tau_2(h_1 \circ g_2, h_1^{\dagger} \circ g_1)\big) \circ \epsilon_{(h_1^{\dagger} \circ g_1)}\big(\xi^{-1}_{(h_1 \circ g_2)}(\tau_2(h_1, g_2))\big) \circ \\
&\quad \xi^{-1}_{(h_1^{\dagger} \circ g_1)}\big(\tau_2(h_1^{\dagger}, g_1)\big) \circ \epsilon_{g_1}\big(\tau_2(h_1, h_1^{\dagger})^{\dagger}\big), \text{ using (3) and (38).} \qquad (41)
\end{aligned}$$

Using (41) in (40), we get

$$\begin{aligned}
&\tau'^{(\tau_1,\tau_2)}\big((h_1, g_1), (h_2, g_2)\big) \\
= \;&\big(\xi^{-1}_{(h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)}(\tau_2(h_1, h_2)),\; \xi^{-1}_{h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1}(\tau_2(h_1 \circ g_2, h_1^{\dagger} \circ g_1)) \circ \\
&\epsilon_{(h_1^{\dagger} \circ g_1)}(\xi^{-1}_{(h_1 \circ g_2)}(\tau_2(h_1, g_2))) \circ \xi^{-1}_{(h_1^{\dagger} \circ g_1)}(\tau_2(h_1^{\dagger}, g_1)) \circ \epsilon_{g_1}(\tau_2(h_1, h_1^{\dagger})^{\dagger})\big) \\
= \;&\big(\xi^{-1}_{(h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)}(\tau_2(h_1, h_2)),\; \xi^{-1}_{h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1}(\tau_2(h_1 \circ g_2, h_1^{\dagger} \circ g_1)) \circ \qquad (42) \\
&\epsilon_{(h_1^{\dagger} \circ g_1)}(\xi^{-1}_{(h_1 \circ g_2)}(\tau_2(h_1, g_2))) \circ \xi^{-1}_{g_1}(\tau_2(h_1, h_1^{\dagger} \circ g_1))^{-1}\big), \\
&\text{using (5) and choosing } m_1 = h_1, m_2 = h_1^{\dagger} \text{and } m_3 = g_1.
\end{aligned}$$

Let $\Upsilon^{-1} : \mathrm{H}^2_{SB}(H, I) \to \mathrm{Ext}_{(\xi,\zeta,\epsilon)}(H, I)$ and $\widetilde{\Upsilon} : \mathrm{Ext}_{(\widetilde{\xi},\widetilde{\zeta},\widetilde{\epsilon})}(\widetilde{H}, \widetilde{I}) \to \mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I})$ be the bijections given by Theorem 2.5. Let $\Theta : \mathrm{Ext}_{(\xi,\zeta,\epsilon)}(H, I) \to \mathrm{Ext}_{(\widetilde{\xi},\widetilde{\zeta},\widetilde{\epsilon})}(\widetilde{H}, \widetilde{I})$ be the map defined in Corollary 4.9. With the preceding set-up, we have the following result.

Proposition 4.10. *Let $(H, \cdot, \circ)$ be a skew left brace and $(I, \cdot)$ be a module over $(H, \cdot, \circ)$ with respect to the action (ξ, ζ, ϵ). Then the map*

$$\Omega_{SB} := \widetilde{\Upsilon}\Theta\Upsilon^{-1} : \mathrm{H}^2_{SB}(H, I) \to \mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I})$$

is a homomorphism of groups.

Proof. The map Ω_{SB} is explicitly given by $\Omega_{SB}([\tau_1, \tau_2]) = [\tau^{(\tau_1,\tau_2)}, \tau'^{(\tau_1,\tau_2)}]$, where $\tau^{(\tau_1,\tau_2)}$ and $\tau'^{(\tau_1,\tau_2)}$ are as described in (39) and (42), respectively. Let $[\tau_1, \tau_2]$ and $[\mu_1, \mu_2]$ be elements in $\mathrm{H}^2_{SB}(H, I)$. Then

$$\Omega_{SB}\big([\tau_1, \tau_2]\,[\mu_1, \mu_2]\big) = \Omega_{SB}\big([\tau_1\mu_1, \tau_2\mu_2]\big) = \big[\tau^{(\tau_1\mu_1,\tau_2\mu_2)}, \tau'^{(\tau_1\mu_1,\tau_2\mu_2)}\big].$$

It is easy to see that

$$\tau^{(\tau_1\mu_1, \tau_2\mu_2)} = \tau^{(\tau_1,\tau_2)}\, \tau^{(\mu_1,\mu_2)} \quad \text{and} \quad \tau'^{(\tau_1\mu_1, \tau_2\mu_2)} = \tau'^{(\tau_1,\tau_2)}\, \tau'^{(\mu_1,\mu_2)}.$$

Hence, we obtain

$$\Omega_{SB}\big([\tau_1,\tau_2]\,[\mu_1,\mu_2]\big) = \big[\tau^{(\tau_1,\tau_2)}, \tau'^{(\tau_1,\tau_2)}\big]\,\big[\tau^{(\mu_1,\mu_2)}, \tau'^{(\mu_1,\mu_2)}\big] = \Omega_{SB}\big([\tau_1,\tau_2]\big)\,\Omega_{SB}\big([\mu_1,\mu_2]\big),$$

which shows that Ω_{SB} is a homomorphism of groups. □

5 Commutative diagram of second cohomology groups

Let $(H, \cdot, \circ)$ be a skew left brace and I an abelian group viewed as a trivial brace. Let $\mathcal{H} = (H^{(\cdot)}, H^{(\circ)}, \lambda^H, \mathrm{id}_H)$ and $\mathcal{I} = (I^{(\cdot)}, I^{(\cdot)}, \lambda^I, \mathrm{id}_I)$, viewed as an $\mathcal{H}-$module with respect to the action (ν, μ, σ, f). By [6, Proposition 4.4 and Corollary 4.6], there is a group isomorphism $\Psi : \mathrm{H}^2_{RRB}(\mathcal{H}, \mathcal{I}) \to \mathrm{H}^2_{SB}(H, I)$ defined by

$$\Psi\big([\tau_1, \tau_2, \rho, \chi]\big) \quad = \quad [\tau_1, \tau_1^{(\lambda^H, \mathrm{id}_H)} \rho^{\mathrm{id}_H} \chi^{(\mathrm{id}_H, f)}], \tag{43}$$

where

$$\begin{aligned} \tau_1^{(\lambda^H, \mathrm{id}_H)}(h_1, h_2) &= \tau_1\big(h_1, \lambda^H_{h_1}(h_2)\big), \\ \rho^{\mathrm{id}_H}(h_1, h_2) &= \rho(h_2, h_1) \text{ and} \\ \chi^{(\mathrm{id}_H, f)}(h_1, h_2) &= \nu_{h_1}\big(f(\chi(h_1), h_2)\big) \end{aligned} \tag{44}$$

for all $h_1, h_2 \in H$. Here, the cohomology $\mathrm{H}^2_{SB}(H, I)$ is with respect to the induced action (ξ, ζ, ϵ), where $\xi = \nu$, $\zeta = \mu$ and $\epsilon = \sigma$.

Similarly, by [14, Proposition 4.8], for the Rota–Baxter group $\widetilde{H}$ and $\widetilde{I}$ viewed as an $\widetilde{H}-$module with associated action γ, there is a group homomorphism

$$\widetilde{\Psi} : \mathrm{H}^2_{RB}(\widetilde{H}, \widetilde{I}) \to \mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I}) \quad \text{defined by} \quad \widetilde{\Psi}\big([\tau, r]\big) = [\tau, \beta],$$

where

$$\begin{aligned} \beta\big((h_1, g_1), (h_2, g_2)\big) = {} & \tau\big((h_1, g_1), i_{\widetilde{R}(h_1, g_1)}(h_2, g_2)\big)\ \tau\big(\widetilde{R}(h_1, g_1) \bullet (h_2, g_2), (\widetilde{R}(h_1, g_1))^{-1}\big) \\ & \gamma_{(\widetilde{R}(h_1, g_1))^{-1}}\big(\tau\big(\widetilde{R}(h_1, g_1), (h_2, g_2)\big)\, \gamma_{(h_2, g_2)}\big(r(h_1, g_1)\big)\ \big(r(h_1, g_1)\big)^{-1}\big) \\ & \big(\tau\big(\widetilde{R}(h_1, g_1), (\widetilde{R}(h_1, g_1))^{-1}\big)\big)^{-1} \end{aligned} \tag{45}$$

and i_x is the inner automorphism given by $i_x(y) = xyx^{-1}$. Here, the cohomology $\mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I})$ is with respect to the induced associated action $(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\epsilon})$. With the preceding set-up, we prove the following result.

Theorem 5.1. *Let $(H, \cdot, \circ)$ be a skew left brace and I an abelian group viewed as a trivial brace. Let $\mathcal{H} = (H^{(\cdot)}, H^{(\circ)}, \lambda^H, \mathrm{id}_H)$ and $\mathcal{I} = (I^{(\cdot)}, I^{(\cdot)}, \lambda^I, \mathrm{id}_I)$ viewed as an $\mathcal{H}-$module with respect to the action (ν, μ, σ, f). Then the diagram*

$$\begin{tikzcd}
\mathrm{H}^2_{RRB}(\mathcal{H}, \mathcal{I}) \arrow[r, "\Omega_{RB}"] \arrow[d, "\Psi"'] & \mathrm{H}^2_{RB}(\widetilde{H}, \widetilde{I}) \arrow[d, "\widetilde{\Psi}"'] \\
\mathrm{H}^2_{SB}(H, I) \arrow[r, "\Omega_{SB}"] & \mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I})
\end{tikzcd} \tag{46}$$

commutes, where H^2_{RB} and H^2_{SB} are the cohomologies with respect to the induced module structures.

Proof. We first establish our notation for clarity. Let $\mathrm{H}^2_{RRB}(\mathcal{H}, \mathcal{I})$ be the second cohomology of $\mathcal{H}$ with coefficients in the module $\mathcal{I}$ with respect to the action (ν, μ, σ, f), where the map f is trivial. For $[\tau_1, \tau_2, \rho, \chi] \in \mathrm{H}^2_{RRB}(\mathcal{H}, \mathcal{I})$, we write

$$\Psi\big([\tau_1, \tau_2, \rho, \chi]\big) = [\tau_1, \hat{\tau_2}] \in \mathrm{H}^2_{SB}(H, I),$$

where the cohomology $\mathrm{H}^2_{SB}(H, I)$ is with respect to the induced action (ξ, ζ, ϵ). Setting $\varphi = (\tau_1, \hat{\tau_2})$, we write

$$\Omega_{SB}\big([\tau_1, \hat{\tau_2}]\big) = [\tau^{\varphi}, \tau'^{\varphi}] \in \mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I}),$$

with the induced associated action $(\tilde{\xi}, \tilde{\zeta}, \tilde{\epsilon})$ as defined in (32), (34) and (35).

In view of Proposition 3.4, for $[\tau_1, \tau_2, \rho, \chi] \in \mathrm{H}^2_{RRB}(\mathcal{H}, \mathcal{I})$, setting $\kappa = (\tau_1, \tau_2, \rho, \chi)$, we write

$$\Omega_{RB}\big([\tau_1, \tau_2, \rho, \chi]\big) = \big[\tau^{\kappa}, r^{\kappa}\big] \in \mathrm{H}^2_{RB}(\widetilde{H}, \widetilde{I}),$$

where the cohomology $\mathrm{H}^2_{RB}(\widetilde{H}, \widetilde{I})$ is with respect to the induced action γ given by (28). Further, for $\big[\tau^{\kappa}, r^{\kappa}\big] \in \mathrm{H}^2_{RB}(\widetilde{H}, \widetilde{I})$, we write

$$\widetilde{\Psi}\big([\tau^{\kappa}, r^{\kappa}]\big) = [\tau^{\kappa}, \beta^{r^{\kappa}}] \in \mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I}),$$

where the cohomology $\mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I})$ is with respect to the induced action $(\widetilde{\xi}, \widetilde{\zeta}, \widetilde{\epsilon})$. It is a direct observation that the induced action defining the cohomology $\mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I})$ is identical from both the directions of the diagram (46).

Let $\widetilde{R}$ be the induced Rota–Baxter operator on $\widetilde{H}$ as given in (31). Using (45), we compute the terms of $\beta^{r^{\kappa}}\big((h_1, g_1), (h_2, g_2)\big)$ individually. First, we have

$$\begin{aligned}
& \tau^{\kappa}\big((h_1, g_1), i_{\widetilde{R}(h_1, g_1)}(h_2, g_2)\big) \\
= \; & \tau^{\kappa}\big((h_1, g_1), (1, g_1^{\dagger} \circ h_1) \bullet (h_2, g_2) \bullet (1, h_1^{\dagger} \circ g_1)\big) \\
= \; & \tau^{\kappa}\big((h_1, g_1), (\lambda^H_{g_1^{\dagger} \circ h_1}(h_2), g_1^{\dagger} \circ h_1 \circ g_2) \bullet (1, h_1^{\dagger} \circ g_1)\big) \\
= \; & \tau^{\kappa}\big((h_1, g_1), (\lambda^H_{g_1^{\dagger} \circ h_1}(h_2), g_1^{\dagger} \circ h_1 \circ g_2 \circ h_1^{\dagger} \circ g_1)\big)
\end{aligned}$$

$$\begin{aligned}
&= \big(\nu^{-1}_{(h_1\circ g_2\circ h_1^\dagger\circ g_1)}\big(\tau_1(h_1, \lambda^H_{g_1}(\lambda^H_{g_1^\dagger\circ h_1}(h_2)))\,\rho(\lambda^H_{g_1^\dagger\circ h_1}(h_2), g_1)\big), \tau_2(g_1, g_1^\dagger\circ h_1\circ g_2\circ h_1^\dagger\circ g_1)\big),\\
&\quad \text{using } (29)\\
&= \big(\nu^{-1}_{(h_1\circ g_2\circ h_1^\dagger\circ g_1)}\big(\tau_1(h_1, \lambda^H_{h_1}(h_2))\,\rho(\lambda^H_{g_1^\dagger\circ h_1}(h_2), g_1)\big), \tau_2(g_1, g_1^\dagger\circ h_1\circ g_2\circ h_1^\dagger\circ g_1)\big). \qquad (47)
\end{aligned}$$

Second, we have

$$\begin{aligned}
&\tau^\kappa\big(\widetilde{R}(h_1, g_1)\bullet(h_2, g_2), (\widetilde{R}(h_1, g_1))^{-1}\big)\\
=\;&\tau^\kappa\big((1, g_1^\dagger\circ h_1)\bullet(h_2, g_2), (1, h_1^\dagger\circ g_1)\big)\\
=\;&\tau^\kappa\big((\lambda^H_{g_1^\dagger\circ h_1}(h_2), g_1^\dagger\circ h_1\circ g_2), (1, h_1^\dagger\circ g_1)\big)\\
=\;&\big(\nu^{-1}_{(g_1^\dagger\circ h_1\circ g_2\circ h_1^\dagger\circ g_1)}\big(\tau_1(\lambda^H_{g_1^\dagger\circ h_1}(h_2), 1)\,\rho(1, g_1^\dagger\circ h_1\circ g_2)\big), \tau_2(g_1^\dagger\circ h_1\circ g_2, h_1^\dagger\circ g_1)\big),\\
&\text{using } (29)\\
=\;&\big(1, \tau_2(g_1^\dagger\circ h_1\circ g_2, h_1^\dagger\circ g_1)\big), \text{ since } \tau_1(\lambda^H_{g_1^\dagger\circ h_1}(h_2), 1) = 1 = \rho(1, g_1^\dagger\circ h_1\circ g_2). \qquad (48)
\end{aligned}$$

Third, we have

$$\begin{aligned}
&\tau^\kappa\big(\widetilde{R}(h_1, g_1), (h_2, g_2)\big)\\
=\;&\tau^\kappa\big((1, g_1^\dagger\circ h_1), (h_2, g_2)\big)\\
=\;&\big(\nu^{-1}_{g_1^\dagger\circ h_1\circ g_2}\big(\tau_1(1, \lambda^H_{g_1^\dagger\circ h_1}(h_2))\,\rho(h_2, g_1^\dagger\circ h_1)\big), \tau_2(g_1^\dagger\circ h_1, g_2)\big), \text{using } (29)\\
=\;&\big(\nu^{-1}_{g_1^\dagger\circ h_1\circ g_2}\big(\rho(h_2, g_1^\dagger\circ h_1)\big), \tau_2(g_1^\dagger\circ h_1, g_2)\big), \text{ since } \tau_1(1, \lambda^H_{g_1^\dagger\circ h_1}(h_2)) = 1. \qquad (49)
\end{aligned}$$

Using (49), we get

$$\begin{aligned}
&\gamma_{(\widetilde{R}(h_1, g_1))^{-1}}\big(\tau^\kappa\big(\widetilde{R}(h_1, g_1), (h_2, g_2)\big)\big)\\
=\;&\gamma_{(1, h_1^\dagger\circ g_1)}\big(\nu^{-1}_{g_1^\dagger\circ h_1\circ g_2}\big(\rho(h_2, g_1^\dagger\circ h_1)\big), \tau_2(g_1^\dagger\circ h_1, g_2)\big)\\
=\;&\big(\nu^{-1}_{h_1^\dagger\circ g_1}\big(\mu_1(\nu^{-1}_{g_1^\dagger\circ h_1\circ g_2}\big(\rho(h_2, g_1^\dagger\circ h_1)\big)\, f(\tau_2(g_1^\dagger\circ h_1, g_2), 1)\big), \sigma_{h_1^\dagger\circ g_1}(\tau_2(g_1^\dagger\circ h_1, g_2))\big),\\
&\text{using } (28)\\
=\;&\big(\nu^{-1}_{h_1^\dagger\circ g_1}(\nu^{-1}_{g_1^\dagger\circ h_1\circ g_2}\big(\rho(h_2, g_1^\dagger\circ h_1))\big), \sigma_{h_1^\dagger\circ g_1}(\tau_2(g_1^\dagger\circ h_1, g_2))\big), \qquad (50)\\
&\text{since } \mu_1 = \mathrm{id}_I \text{ and } f(\tau_2(g_1^\dagger\circ h_1, g_2), 1) = 1.
\end{aligned}$$

Considering the remaining three terms, we have

$$\begin{aligned}
&\gamma_{(\widetilde{R}(h_1, g_1))^{-1}}\big(\gamma_{(h_2, g_2)} r^\kappa(h_1, g_1)\big)\\
=\;&\gamma_{(1, h_1^\dagger\circ g_1)}\big(\gamma_{(h_2, g_2)}\big(1, \tau_2(g_1, g_1^\dagger\circ h_1)^{-1}\chi(h_1)\big)\big), \text{ using } (30)\\
=\;&\gamma_{(1, h_1^\dagger\circ g_1)}\big(\nu^{-1}_{g_2}\big(\mu_{h_2}(1) f(\tau_2(g_1, g_1^\dagger\circ h_1)^{-1}\chi(h_1), h_2)\big), \sigma_{g_2}(\tau_2(g_1, g_1^\dagger\circ h_1)^{-1}\chi(h_1))\big),\\
&\text{using } (28)\\
=\;&\gamma_{(1, h_1^\dagger\circ g_1)}\big(\nu^{-1}_{g_2}\big(f(\tau_2(g_1, g_1^\dagger\circ h_1)^{-1}\chi(h_1), h_2)\big), \sigma_{g_2}(\tau_2(g_1, g_1^\dagger\circ h_1)^{-1}\chi(h_1))\big)
\end{aligned}$$

$$= \big(\nu^{-1}_{g_2\circ h_1^\dagger\circ g_1}(f(\tau_2(g_1, g_1^\dagger\circ h_1)^{-1}\chi(h_1), h_2)), \sigma_{g_2\circ h_1^\dagger\circ g_1}(\tau_2(g_1, g_1^\dagger\circ h_1)^{-1}\chi(h_1))\big), \tag{51}$$

using (28).

Using (28) and (30), we have

$$\gamma_{(\widetilde{R}(h_1,g_1))^{-1}}\big((r^\kappa(h_1,g_1))^{-1}\big) = \big(1, \sigma_{h_1^\dagger\circ g_1}(\chi(h_1)^{-1}\tau_2(g_1, g_1^\dagger\circ h_1))\big), \tag{52}$$

and

$$\begin{aligned}
&\tau^\kappa\big(\widetilde{R}(h_1,g_1), (\widetilde{R}(h_1,g_1))^{-1}\big)^{-1}\\
=\;&\tau^\kappa\big((1, g_1^\dagger\circ h_1), (1, h_1^\dagger\circ g_1)\big)^{-1}\\
=\;&\big(1, \tau_2(g_1^\dagger\circ h_1, h_1^\dagger\circ g_1)\big)^{-1}, \text{ using (29)}\\
=\;&\big(1, \tau_2(g_1^\dagger\circ h_1, h_1^\dagger\circ g_1)^{-1}\big).
\end{aligned} \tag{53}$$

Using (47), (48), (50), (51), (52) and (53), we can write

$$\beta^{r^\kappa}\big((h_1,g_1),(h_2,g_2)\big) = \big(\beta_1^{r^\kappa}\big((h_1,g_1),(h_2,g_2)\big),\ \beta_2^{r^\kappa}\big((h_1,g_1),(h_2,g_2)\big)\big),$$

where

$$\begin{aligned}
\beta_1^{r^\kappa}\big((h_1,g_1),(h_2,g_2)\big) \;=\;& \nu_{(g_1^\dagger\circ h_1\circ g_2^\dagger\circ h_1^\dagger)}\big(\tau_1(h_1, \lambda^H_{h_1}(h_2))\,\rho(\lambda^H_{g_1^\dagger\circ h_1}(h_2), g_1)\big)\\
&\nu_{(g_1^\dagger\circ h_1\circ g_2^\dagger\circ h_1^\dagger)}\big(\nu_{g_1}(\rho(h_2, g_1^\dagger\circ h_1))\big)\\
&\nu^{-1}_{g_2\circ h_1^\dagger\circ g_1}\big(f(\tau_2(g_1, g_1^\dagger\circ h_1)^{-1}\chi(h_1), h_2)\big)
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
\beta_2^{r^\kappa}\big((h_1,g_1),(h_2,g_2)\big) \;=\;& \tau_2(g_1, g_1^\dagger\circ h_1\circ g_2\circ h_1^\dagger\circ g_1)\,\tau_2(g_1^\dagger\circ h_1\circ g_2, h_1^\dagger\circ g_1)\\
&\sigma_{h_1^\dagger\circ g_1}\big(\tau_2(g_1^\dagger\circ h_1, g_2)\big)\,\sigma_{g_2\circ h_1^\dagger\circ g_1}\big(\tau_2(g_1, g_1^\dagger\circ h_1)^{-1}\chi(h_1)\big)\\
&\sigma_{h_1^\dagger\circ g_1}\big(\chi(h_1)^{-1}\tau_2(g_1, g_1^\dagger\circ h_1)\big)\,\tau_2(g_1^\dagger\circ h_1, h_1^\dagger\circ g_1)^{-1}.
\end{aligned}$$

We now simplify these expressions. Using appropriate substitutions, we get

$$\begin{aligned}
&\nu_{(g_1^\dagger\circ h_1\circ g_2^\dagger\circ h_1^\dagger)}\big(\rho(\lambda^H_{g_1^\dagger\circ h_1}(h_2), g_1)\,\nu_{g_1}(\rho(h_2, g_1^\dagger\circ h_1))\big)\\
=\;&\nu_{(g_1^\dagger\circ h_1\circ g_2^\dagger\circ h_1^\dagger)}\big(\rho(h_2, h_1)\,\nu_{h_1}(f(\tau_2(g_1, g_1^\dagger\circ h_1), h_2))\big), \text{ using (23)}\\
=\;&\nu_{(g_1^\dagger\circ h_1\circ g_2^\dagger\circ h_1^\dagger)}\big(\rho(h_2, h_1)\big)\nu^{-1}_{g_2\circ h_1^\dagger\circ g_1}\big((f(\tau_2(g_1, g_1^\dagger\circ h_1), h_2))\big).
\end{aligned}$$

Using the preceding equality in (54), we get

$$\beta_1^{r^\kappa}\big((h_1,g_1),(h_2,g_2)\big)$$

$$= \quad \nu_{(g_1^\dagger \circ h_1 \circ g_2^\dagger \circ h_1^\dagger)}\big(\tau_1(h_1, \lambda^H_{h_1}(h_2))\rho(h_2, h_1)\big)\, \nu_{g_1^\dagger \circ h_1 \circ g_2^\dagger}\big(f(\chi(h_1), h_2)\big), \tag{55}$$

using Lemma (2.19)(2).

For the expression $\beta_2^{r^\kappa}$, we have

$$\begin{aligned}
&\beta_2^{r^\kappa}\big((h_1, g_1), (h_2, g_2)\big) \\
= \;& \tau_2(g_1, g_1^\dagger \circ h_1 \circ g_2 \circ h_1^\dagger \circ g_1)\, \tau_2(g_1^\dagger \circ h_1 \circ g_2, h_1^\dagger \circ g_1) \\
& \sigma_{h_1^\dagger \circ g_1}\big(\tau_2(g_1^\dagger \circ h_1, g_2)\big)\, \sigma_{g_2 \circ h_1^\dagger \circ g_1}\big(\tau_2(g_1, g_1^\dagger \circ h_1)^{-1}\chi(h_1)\big) \\
& \sigma_{h_1^\dagger \circ g_1}\big(\chi(h_1)^{-1}\tau_2(g_1, g_1^\dagger \circ h_1)\big)\, \tau_2(g_1^\dagger \circ h_1, h_1^\dagger \circ g_1)^{-1} \\
= \;& \tau_2(h_1 \circ g_2, h_1^\dagger \circ g_1)\, \sigma_{h_1^\dagger \circ g_1}\big(\tau_2(g_1, g_1^\dagger \circ h_1 \circ g_2)\big) \\
& \sigma_{h_1^\dagger \circ g_1}\big(\tau_2(g_1^\dagger \circ h_1, g_2)\big)\, \sigma_{g_2 \circ h_1^\dagger \circ g_1}\big(\tau_2(g_1, g_1^\dagger \circ h_1)^{-1}\chi(h_1)\big) \\
& \sigma_{h_1^\dagger \circ g_1}\big(\chi(h_1)^{-1}\, \tau_2(g_1, g_1^\dagger \circ h_1)\big)\, \tau_2(g_1^\dagger \circ h_1, h_1^\dagger \circ g_1)^{-1}, \\
& \text{using (22) with appropriate substitution} \\
= \;& \tau_2(h_1 \circ g_2, h_1^\dagger \circ g_1)\, \sigma_{h_1^\dagger \circ g_1}\big(\tau_2(h_1, g_2)\big) \\
& \sigma_{h_1^\dagger \circ g_1}\big(\sigma_{g_2}(\chi(h_1))\chi(h_1)^{-1}\big)\, \tau_2(h_1, h_1^\dagger \circ g_1)^{-1},
\end{aligned} \tag{56}$$

using (22) twice with appropriate substitutions.

Note that, for $[\tau_1, \tau_2, \rho, \chi] \in \mathrm{H}^2_{RRB}(\mathcal{H}, \mathcal{I})$, we have $[\tau^\kappa, \beta^{r^\kappa}] = \widetilde{\Psi}\, \Omega_{RB}\big([\tau_1, \tau_2, \rho, \chi]\big)$ and $[\tau^\varphi, \tau'^\varphi] = \Omega_{SB}\, \Psi\big([\tau_1, \tau_2, \rho, \chi]\big)$. If $[\tau^\varphi, \tau'^\varphi]^{-1}$ denotes the inverse of $[\tau^\varphi, \tau'^\varphi]$ in $\mathrm{H}^2_{SB}(\widetilde{H}, \widetilde{I})$, then we have

$$[\tau^\kappa, \beta^{r^\kappa}][\tau^\varphi, \tau'^\varphi]^{-1} = [\tau^\kappa(\tau^\varphi)^{-1}, \beta^{r^\kappa}(\tau'^\varphi)^{-1}]. \tag{57}$$

We claim that $(\tau^\kappa(\tau^\varphi)^{-1},\, \beta^{r^\kappa}(\tau'^\varphi)^{-1}) \in \mathrm{Z}^2_{SB}(\widetilde{H}, \widetilde{I})$. First, we see that

$$\begin{aligned}
&\tau^\kappa\big((h_1, g_1), (h_2, g_2)\big)\big(\tau^\varphi((h_1, g_1), (h_2, g_2))\big)^{-1} \\
&\quad = \big(\nu^{-1}_{g_1 \circ g_2}\big(\tau_1(h_1, \lambda^H_{g_1}(h_2))\rho(h_2, g_1)\big), \tau_2(g_1, g_2)\big) \\
&\quad\quad \big(\xi^{-1}_{(g_1 \circ g_2)}\big(\tau_1(h_1, \lambda^H_{g_1}(h_2))\tau_1(g_1^{-1}, g_1 \circ h_2)\zeta_{(g_1 \circ h_2)}(\tau_1(g_1, g_1^{-1})^{-1})\hat{\tau_2}(g_1, h_2)\big), \\
&\quad\quad\quad \xi^{-1}_{(g_1 \circ g_2)}(\hat{\tau_2}(g_1, g_2))\big)^{-1}, \text{using (29) and (39)},
\end{aligned}$$

for all $h_1, h_2, g_1, g_2 \in H$. By (43), we have

$$\hat{\tau_2}(g_1, h_2) = \tau_1(g_1, \lambda^H_{g_1}(h_2))\, \rho(h_2, g_1)\, \nu_{g_1}(f(\chi(g_1), h_2)). \tag{58}$$

Using (21) with appropriate substitutions, we get

$$\tau_1(g_1^{-1}, g_1 \circ h_2)\, \mu_{(g_1 \circ h_2)}(\tau_1(g_1, g_1^{-1})^{-1})\, \tau_1(g_1, \lambda^H_{g_1}(h_2)) = 1 \tag{59}$$

By [6, Corollary 4.2 and Proposition 4.3], we have $\xi_h = \nu_h$, $\zeta_h = \mu_h$ and $\sigma_h = \epsilon_h$ for all $h \in H$. This gives

$$\big(\nu^{-1}_{g_1 \circ g_2}\big(\tau_1(h_1, \lambda^H_{g_1}(h_2))\rho(h_2, g_1)\big)\big)$$

$$\begin{aligned}
&\big(\xi^{-1}_{(g_1\circ g_2)}(\tau_1(h_1,\lambda^H_{g_1}(h_2))\tau_1(g_1^{-1},g_1\circ h_2)\zeta_{(g_1\circ h_2)}(\tau_1(g_1,g_1^{-1})^{-1})\hat{\tau}_2(g_1,h_2))\big)^{-1}\\
&=\big(\nu^{-1}_{g_1\circ g_2}\big(\tau_1(h_1,\lambda^H_{g_1}(h_2))\rho(h_2,g_1)\big)\big)\\
&\quad\big(\nu^{-1}_{(g_1\circ g_2)}(\tau_1(h_1,\lambda^H_{g_1}(h_2))\tau_1(g_1^{-1},g_1\circ h_2)\mu_{(g_1\circ h_2)}(\tau_1(g_1,g_1^{-1})^{-1})\hat{\tau}_2(g_1,h_2))\big)^{-1}\\
&=\big(\nu_{g_2^\dagger}(f(\chi(g_1),h_2))\big)^{-1},\ \text{using (58) and (59).}
\end{aligned}$$

Similarly, using (58) and (25), we get

$$\tau_2(g_1,g_2)\,\big(\xi^{-1}_{(g_1\circ g_2)}(\hat{\tau}_2(g_1,g_2))\big)^{-1}=\chi(g_2)^{-1}\,\chi(g_1\circ g_2)\,\sigma_{g_2}(\chi(g_1)^{-1}). \tag{60}$$

Thus, we have

$$\begin{aligned}
&\tau^{\kappa}\big((h_1,g_1),(h_2,g_2)\big)\,\big(\tau^{\varphi}((h_1,g_1),(h_2,g_2))\big)^{-1}\\
=\;&\big(\big(\nu_{g_2^\dagger}(f(\chi(g_1),h_2))\big)^{-1},\,\chi(g_2)^{-1}\chi(g_1\circ g_2)\sigma_{g_2}(\chi(g_1)^{-1})\big).
\end{aligned}\tag{61}$$

Similarly, using (42), (55) and (56), we see that:

$$\beta^{r^{\kappa}}\,((h_1,g_1),\,(h_2,g_2))\;(\tau'^{\varphi}((h_1,g_1),(h_2,g_2)))^{-1}=(h,g),$$

where

$$\begin{aligned}
h=\;&\nu_{(g_1^\dagger\circ h_1\circ g_2^\dagger\circ h_1^\dagger)}\,(\tau_1(h_1,\lambda^H_{h_1}(h_2))\rho(h_2,h_1))\\
&\nu_{g_1^\dagger\circ h_1\circ g_2^\dagger}(f(\chi(h_1),h_2))\,\Big(\xi^{-1}_{(h_1\circ g_2\circ h_1^\dagger\circ g_1)}(\hat{\tau}_2(h_1,h_2))\Big)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
g=\;&\tau_2(h_1\circ g_2,h_1^\dagger\circ g_1)\,\sigma_{h_1^\dagger\circ g_1}(\tau_2(h_1,g_2))\,\sigma_{h_1^\dagger\circ g_1}(\sigma_{g_2}(\chi(h_1))\chi(h_1)^{-1})\,\tau_2(h_1,h_1^\dagger\circ g_1)^{-1}\\
&\xi^{-1}_{h_1\circ g_2\circ h_1^\dagger\circ g_1}(\hat{\tau}_2(h_1\circ g_2,h_1^\dagger\circ g_1))\circ\epsilon_{(h_1^\dagger\circ g_1)}(\xi^{-1}_{(h_1\circ g_2)}(\hat{\tau}_2(h_1,g_2)))\circ\xi^{-1}_{g_1}(\hat{\tau}_2(h_1,h_1^\dagger\circ g_1))^{-1}\big)^{-1}.
\end{aligned}$$

Using (58), and the fact that $\xi_h=\nu_h$ for all $h\in H$, the first component becomes trivial. Further, by multiple use of (60), we obtain

$$\begin{aligned}
&\beta^{r^{\kappa}}\big((h_1,g_1),(h_2,g_2)\big)\,\big(\tau'^{\varphi}((h_1,g_1),(h_2,g_2))\big)^{-1}\\
=\;&\big(1,\,\chi(g_1)^{-1}\chi(h_1\circ g_2\circ h_1^\dagger\circ g_1)\,\sigma_{h_1^\dagger\circ g_1}(\chi(g_2))^{-1}\big).
\end{aligned}\tag{62}$$

Consider the map $\theta:\widetilde{H}\to I\times I$ given by

$$\theta(h,g)=\big(1,\chi(g)^{-1}\big)$$

for all $h,g\in H$. In view of (7) and (8), to prove the claim, it suffices to show that

$$\tau^{\kappa}\big((h_1,g_1),(h_2,g_2)\big)\,\big(\tau^{\varphi}((h_1,g_1),(h_2,g_2))\big)^{-1}$$

$$= \theta\big((h_1, g_1) \bullet (h_2, g_2)\big)^{-1}\widetilde{\zeta}_{(h_2,g_2)}\big(\theta(h_1, g_1)\big)\, \theta(h_2, g_2)$$

and

$$\begin{aligned}
&\beta^{r^\kappa}\big((h_1, g_1), (h_2, g_2)\big)\, \big(\tau'^{\varphi}((h_1, g_1), (h_2, g_2))\big)^{-1} \\
&\quad = \theta\big((h_1, g_1) \odot (h_2, g_2)\big)^{-1}\, \widetilde{\xi}_{(h_1,g_1)\odot(h_2,g_2)}\big(\widetilde{\epsilon}_{(h_2,g_2)}(\widetilde{\xi}^{-1}_{(h_1,g_1)}(\theta(h_1, g_1)))\big)\widetilde{\xi}_{(h_1,g_1)}\big(\theta(h_2, g_2)\big)
\end{aligned}$$

for all $h_1, g_1, h_2, g_2 \in H$. First, note that

$$\theta\big((h_1, g_1) \bullet (h_2, g_2)\big)^{-1} = \theta\big(h_1 \lambda^H_{g_1}(h_2),\ g_1 \circ g_2\big)^{-1} = \big(1,\ \chi(g_1 \circ g_2)^{-1}\big)^{-1} = \big(1,\ \chi(g_1 \circ g_2)\big)$$

and

$$\begin{aligned}
\widetilde{\zeta}_{(h_2,g_2)}\big(\theta(h_1, g_1)\big) &= \widetilde{\zeta}_{(h_2,g_2)}\big(1, \chi(g_1)^{-1}\big) \\
&= \big(\xi^{-1}_{g_2}(\zeta_{h_2}(\chi(g_1))\xi_{h_2}(\epsilon_{h_2}(\chi(g_1)^{-1}))),\ \epsilon_{g_2}(\chi(g_1)^{-1})\big), \text{ using (34).}
\end{aligned}$$

By [6, Corollary 4.2 and Proposition 4.3], $\nu_h = \xi_h$ and $f(k, h) = \zeta_h(k^{-1})\xi_h(\epsilon_h(k))$, for all $h \in H$ and $k \in I$. Therefore,

$$\begin{aligned}
\widetilde{\zeta}_{(h_2,g_2)}\big(\theta(h_1, g_1)\big) &= \big(\nu_{g_2^\dagger}(f(\chi(g_1)^{-1}, h_2)),\ \epsilon_{g_2}(\chi(g_1)^{-1})\big) \\
&= \big(\big(\nu_{g_2^\dagger}(f(\chi(g_1), h_2))\big)^{-1},\ \epsilon_{g_2}(\chi(g_1)^{-1})\big), \text{ using Lemma (2.19)(2).}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\theta\big((h_1, g_1) \bullet (h_2, g_2)\big)^{-1}\, \widetilde{\zeta}_{(h_2,g_2)}\big(\theta(h_1, g_1)\big)\, \theta(h_2, g_2) \\
= \;& \big(1,\ \chi(g_1 \circ g_2)\big)\big(\big(\nu_{g_2^\dagger}(f(\chi(g_1), h_2))\big)^{-1},\ \epsilon_{g_2}(\chi(g_1)^{-1})\big)\big(1,\ \chi(g_2)^{-1}\big) \\
= \;& \big(\big(\nu_{g_2^\dagger}(f(\chi(g_1), h_2))\big)^{-1},\ \chi(g_1 \circ g_2)\epsilon_{g_2}(\chi(g_1)^{-1})\chi(g_2)^{-1}\big) \\
= \;& \big(\big(\nu_{g_2^\dagger}(f(\chi(g_1), h_2))\big)^{-1},\ \chi(g_1 \circ g_2)\sigma_{g_2}(\chi(g_1)^{-1})\chi(g_2)^{-1}\big), \text{ since } \sigma_h = \epsilon_h \text{ for all } h \in H \\
= \;& \tau^\kappa\big((h_1, g_1), (h_2, g_2)\big)\, \big(\tau^\varphi((h_1, g_1), (h_2, g_2))\big)^{-1}, \text{ by (61).}
\end{aligned}$$

Next, note that

$$\theta\big((h_1, g_1) \odot (h_2, g_2)\big)^{-1} = \theta(h_1 \circ h_2, h_1 \circ g_2 \circ h_1^\dagger \circ g_1)^{-1} = \big(1,\ \chi(h_1 \circ g_2 \circ h_1^\dagger \circ g_1)\big) \tag{63}$$

and

$$\widetilde{\xi}^{-1}_{(h_1,g_1)}\big(\theta(h_1, g_1)\big) = \widetilde{\xi}_{(h_1^\dagger, h_1^\dagger \circ g_1^\dagger \circ h_1)}\big(1,\ \chi(g_1)^{-1}\big) = \big(1,\ \epsilon_{g_1^\dagger \circ h_1}(\chi(g_1)^{-1})\big), \quad \text{using (32).}$$

This gives

$$\begin{aligned}
&\widetilde{\xi}_{(h_1,g_1)\odot(h_2,g_2)}(\widetilde{\epsilon}_{(h_2,g_2)}(\widetilde{\xi}^{-1}_{(h_1,g_1)}(\theta(h_1, g_1)))) \\
= \;& \widetilde{\xi}_{(h_1 \circ h_2, h_1 \circ g_2 \circ h_1^\dagger \circ g_1)}(\widetilde{\epsilon}_{(h_2,g_2)}(1,\ \epsilon_{g_1^\dagger \circ h_1}(\chi(g_1)^{-1})))
\end{aligned}$$

$$\begin{aligned} &= \big(1,\ \epsilon_{h_2^\dagger\circ g_2\circ h_1^\dagger\circ g_1}(\epsilon_{h_2}(\epsilon_{g_2}^{-1}(\epsilon_{g_1^\dagger\circ h_1}(\chi(g_1)^{-1}))))\big),\ \text{using (32) and (35).}\\ &= \big(1,\ \chi(g_1)^{-1}\big),\ \text{since } \epsilon : H^{(\circ)} \to \operatorname{Aut}(I) \text{ is an anti-homomorphism.} \qquad (64)\end{aligned}$$

Further, we have

$$\widetilde{\xi}_{(h_1,g_1)}\big(\theta(h_2,g_2)\big) = \widetilde{\xi}_{(h_1,g_1)}\big(1,\ \chi(g_2)^{-1}\big) = \big(1,\ \epsilon_{h_1^\dagger\circ g_1}(\chi(g_2)^{-1})\big),\ \text{using (32).} \qquad (65)$$

Using (63), (64) and (65), we obtain

$$\begin{aligned} &\theta\big((h_1,g_1)\odot(h_2,g_2)\big)^{-1}\ \widetilde{\xi}_{(h_1,g_1)\odot(h_2,g_2)}\big(\widetilde{\epsilon}_{(h_2,g_2)}(\widetilde{\xi}^{-1}_{(h_1,g_1)}(\theta(h_1,g_1)))\big)\ \widetilde{\xi}_{(h_1,g_1)}\big(\theta(h_2,g_2)\big)\\ &= \big(1,\ \chi(h_1\circ g_2\circ h_1^\dagger\circ g_1)\big)\big(1,\ \chi(g_1)^{-1}\big)\big(1,\ \epsilon_{h_1^\dagger\circ g_1}(\chi(g_2)^{-1})\big)\\ &= \big(1,\ \chi(h_1\circ g_2\circ h_1^\dagger\circ g_1)\chi(g_1)^{-1}\epsilon_{h_1^\dagger\circ g_1}(\chi(g_2)^{-1})\big)\\ &= \big(1,\ \chi(h_1\circ g_2\circ h_1^\dagger\circ g_1)\chi(g_1)^{-1}\sigma_{h_1^\dagger\circ g_1}(\chi(g_2)^{-1})\big),\ \text{since } \sigma_h = \epsilon_h \text{ for all } h\in H\\ &= \beta^{r^\kappa}\big((h_1,g_1),(h_2,g_2)\big)\ \big(\tau'^{\varphi}((h_1,g_1),(h_2,g_2))\big)^{-1},\ \text{by (62).}\end{aligned}$$

This proves our claim, and hence the proof of the theorem is complete. ☐

6 Isoclinism of squares of skew left braces

Recall from [9, Definition 6, 7] that the *annihilator* $\operatorname{Ann}(H)$ of a skew left brace $(H,\cdot,\circ)$ is defined as

$$\operatorname{Ann}(H) = \ker(\lambda^H)\cap \mathrm{Z}(H^{(\cdot)})\cap \operatorname{Fix}(\lambda^H) = \{a\in H \mid b\circ a = a\circ b = b\cdot a = a\cdot b \text{ for all } b\in H\}.$$

Similarly, the *commutator* H' of a skew left brace $(H,\cdot,\circ)$ is defined as the subgroup of $H^{(\cdot)}$ generated by the commutator subgroup of $H^{(\cdot)}$ and the set (see [21, Definition 2.1])

$$\{a^{-1}\cdot(a\circ b)\cdot b^{-1} \mid a,b\in H\}.$$

It turns out that both the annihilator $\operatorname{Ann}(H)$ and the commutator H' are ideals of $(H,\cdot,\circ)$ (see [9, p.4] and [21, p.2892], respectively). We now recall the definition of isoclinism of skew left braces from [21, Definition 2.7].

Definition 6.1. Two skew left braces A and B are said to be isoclinic if there exist skew brace isomorphisms $\xi_1 : A/\operatorname{Ann}(A)\to B/\operatorname{Ann}(B)$ and $\xi_2 : A'\to B'$ such that the diagram

$$\begin{tikzcd} A' \arrow[d, "\xi_2"'] & \big(A/\operatorname{Ann}(A)\big)\times\big(A/\operatorname{Ann}(A)\big) \arrow[l, "\theta_A"'] \arrow[r, "\theta_A^*"] \arrow[d, "\xi_1\times\xi_1"'] & A' \arrow[d, "\xi_2"'] \\ B' & \big(B/\operatorname{Ann}(B)\big)\times\big(B/\operatorname{Ann}(B)\big) \arrow[l, "\theta_B"'] \arrow[r, "\theta_B^*"] & B' \end{tikzcd} \qquad (66)$$

commutes. Here, the maps $\theta_A, \theta_A^* : (A/\operatorname{Ann}(A))\times(A/\operatorname{Ann}(A))\to A'$ are defined by

$$\theta_A(\overline{a},\overline{b}) = a\cdot b\cdot a^{-1}\cdot b^{-1} \text{ and } \theta_A^*(\overline{a},\overline{b}) = \lambda_a^H(b)\cdot b^{-1}$$

for all $a,b\in A$. The maps θ_B and θ_B^* are defined similarly.

The following results from [22, Proposition 3.4 and Theorem 3.7] will be used in the sequel.

Proposition 6.2. *Let* $(H, \cdot, \circ)$ *be a skew left brace and* $(\widetilde{H}, \bullet, \odot)$ *be its square. Then the following assertions hold:*

1. $\mathrm{Z}(\widetilde{H}^{(\bullet)}) \trianglelefteq \mathrm{Fix}(\lambda^H) \times \mathrm{Z}(H^{(\circ)})$.
2. $(\widetilde{H}^{(\bullet)})' = H' \rtimes_{\lambda^H} (H^{(\circ)})'$.

The following observations are direct.

Lemma 6.3. *Let* $(H, \cdot, \circ)$ *be a skew left brace and* $(\widetilde{H}, \bullet, \odot)$ *be its square. Then the following assertions hold:*

1. $\mathrm{Ann}(H) \times \mathrm{Ann}(H) \subseteq \mathrm{Ann}(\widetilde{H})$.
2. $\mathrm{Ann}(H) \times \mathrm{Ann}(H)$ *is an ideal of* $(\widetilde{H}, \bullet, \odot)$.

Lemma 6.4. *Let* $(H, \cdot, \circ)$ *be a skew left brace and* $(\widetilde{H}, \bullet, \odot)$ *be its square. Then the following assertions hold:*

1. $\mathrm{Ann}(\widetilde{H}) \leq \mathrm{Fix}(\lambda^H) \times \mathrm{Z}(H^{(\circ)})$.
2. $\widetilde{H}/\big(\mathrm{Ann}(H) \times \mathrm{Ann}(H)\big) \cong \big(\widetilde{H/\mathrm{Ann}(H)}\big)$.

Proof. By definition, we have $\mathrm{Ann}(\widetilde{H}) = \ker(\lambda^{\widetilde{H}}) \cap \mathrm{Z}(\widetilde{H}^{(\bullet)}) \cap \mathrm{Fix}(\lambda^{\widetilde{H}})$. By Proposition 6.2(1), we have $\mathrm{Z}(\widetilde{H}^{(\bullet)}) \trianglelefteq \mathrm{Fix}(\lambda^H) \times \mathrm{Z}(H^{(\circ)})$, and assertion (1) follows.

Let $\overline{\lambda}$ be the induced action for $H/\mathrm{Ann}(H)$. It is easy to see that the map

$$\phi : \widetilde{H}/\big(\mathrm{Ann}(H) \times \mathrm{Ann}(H)\big) \to \big(\widetilde{H/\mathrm{Ann}(H)}\big),$$

given by $\phi(\overline{(a, b)}) = (\overline{a}, \overline{b})$ for all $a, b \in H$, is an isomorphism of skew left braces [22, Remark 3.6]. □

Proposition 6.5. *Let* $(A, \cdot, \circ)$ *and* $(B, \cdot, \circ)$ *be isoclinic skew left braces via an isoclinism* (ξ_1, ξ_2)*. Then the following assertions hold:*

1. *There exists a skew brace isomorphism*
$$\widetilde{\xi_1} : \widetilde{A}/\big(\mathrm{Ann}(A) \times \mathrm{Ann}(A)\big) \to \widetilde{B}/\big(\mathrm{Ann}(B) \times \mathrm{Ann}(B)\big),$$
defined as $\widetilde{\xi_1}\big(\overline{a}_1, \overline{a}_2\big) = \big(\xi_1(\overline{a}_1), \xi_1(\overline{a}_2)\big)$.
2. *There exists a skew brace isomorphism*
$$\widetilde{\xi_2} : (\widetilde{A})' \to (\widetilde{B})',$$
defined as $\widetilde{\xi_2}(a_1, a_2) = (\xi_2(a_1), \xi_2(a_2))$.

Proof. In view of Lemma 6.4(2), we identify $\widetilde{A}/(\operatorname{Ann}(A)\times\operatorname{Ann}(A))$ with $\big(\widetilde{A/\operatorname{Ann}(A)}\big)$, and similarly for B. It now follows that the map $\widetilde{\xi_1}(\overline{a}_1,\overline{a}_2)=(\xi_1(\overline{a}_1),\xi_1(\overline{a}_2))$ is an isomorphism.

By Proposition 6.2(2), we have $(\widetilde{A}^{(\bullet)})'=A'\rtimes_{\lambda^A}(A^{(\circ)})'$ and $(\widetilde{B}^{(\bullet)})'=B'\rtimes_{\lambda^B}(B^{(\circ)})'$. By [22, Theorem 3.13], the isomorphism $\xi_2:A'\to B'$ induces an isomorphism of groups $\widetilde{\xi_2}:(\widetilde{A}^{(\bullet)})'\to(\widetilde{B}^{(\bullet)})'$, given by $\widetilde{\xi_2}(a_1,a_2)=(\xi_2(a_1),\xi_2(a_2))$. A direct check shows that $\widetilde{\xi_2}$ restricts to an isomorphism $(\widetilde{A})'\to(\widetilde{B})'$. ◻

Theorem 6.6. *Let $(A,\cdot,\circ)$ and $(B,\cdot,\circ)$ be isoclinic skew left braces. If*

$$\operatorname{Ann}(\widetilde{A})=\operatorname{Ann}(A)\times\operatorname{Ann}(A)\quad\textit{and}\quad\operatorname{Ann}(\widetilde{B})=\operatorname{Ann}(B)\times\operatorname{Ann}(B),$$

then $(\widetilde{A},\bullet,\odot)$ and $(\widetilde{B},\bullet,\odot)$ are also isoclinic.

Proof. Let (ξ_1,ξ_2) be an isoclinism from $(A,\cdot,\circ)$ onto $(B,\cdot,\circ)$. We claim that the following diagram commutes

$$\begin{tikzcd}
(\widetilde{A})' \arrow[d, "\widetilde{\xi_2}"'] & \big(\widetilde{A}/\operatorname{Ann}(\widetilde{A})\big)\times\big(\widetilde{A}/\operatorname{Ann}(\widetilde{A})\big) \arrow[l, "\theta_{\widetilde{A}}"'] \arrow[r, "\theta^*_{\widetilde{A}}"] \arrow[d, "\widetilde{\xi_1}\times\widetilde{\xi_1}"'] & (\widetilde{A})' \arrow[d, "\widetilde{\xi_2}"'] \\
(\widetilde{B})' & \big(\widetilde{B}/\operatorname{Ann}(\widetilde{B})\big)\times\big(\widetilde{B}/\operatorname{Ann}(\widetilde{B})\big) \arrow[l, "\theta_{\widetilde{B}}"'] \arrow[r, "\theta^*_{\widetilde{B}}"] & (\widetilde{B})',
\end{tikzcd}\tag{67}$$

where the maps $\theta_{\widetilde{A}},\theta^*_{\widetilde{A}}:\big(\widetilde{A}/\operatorname{Ann}(\widetilde{A})\big)\times\big(\widetilde{A}/\operatorname{Ann}(\widetilde{A})\big)\to(\widetilde{A})'$ are defined by

$$\theta_{\widetilde{A}}\big((\overline{a}_1,\overline{b}_1),(\overline{a}_2,\overline{b}_2)\big)=(a_1,b_1)\bullet(a_2,b_2)\bullet(a_1,b_1)^{-1}\bullet(a_2,b_2)^{-1}$$

and

$$\theta^*_{\widetilde{A}}\big((\overline{a}_1,\overline{b}_1),(\overline{a}_2,\overline{b}_2)\big)=(1,b_1^{\dagger}\circ a_1)\bullet(a_2,b_2)\bullet(1,b_1^{\dagger}\circ a_1)^{-1}\bullet(a_2,b_2)^{-1}$$

for all $a_1,b_1,a_2,b_2\in A$. The maps $\theta_{\widetilde{B}}$ and $\theta^*_{\widetilde{B}}$ are defined similarly.

First, we prove the commutativity of the left hand side of the diagram. To avoid complexity of the notation, we will denote the composition $A\to A/\operatorname{Ann}(A)\to B/\operatorname{Ann}(B)$, where $A\to A/\operatorname{Ann}(A)$ is the canonical map, by ξ_1. We see that

$$\begin{aligned}
\theta_{\widetilde{A}}\big((\overline{a}_1,\overline{b}_1),(\overline{a}_2,\overline{b}_2)\big) &= (a_1,b_1)\bullet(a_2,b_2)\bullet(a_1,b_1)^{-1}\bullet(a_2,b_2)^{-1}\\
&= \big(a_1\lambda^A_{b_1}(a_2),\, b_1\circ b_2\big)\bullet\big(\lambda^A_{b_1^{\dagger}}(a_1)^{-1},\, b_1^{\dagger}\big)\bullet\big(\lambda^A_{b_2^{\dagger}}(a_2)^{-1},\, b_2^{\dagger}\big)\\
&= \big(a_1\lambda^A_{b_1}(a_2),\, b_1\circ b_2\big)\bullet\big(\lambda^A_{b_1^{\dagger}}(a_1)^{-1}\lambda^A_{b_1}(\lambda^A_{b_2^{\dagger}}(a_2)^{-1}),\, b_1^{\dagger}\circ b_2^{\dagger}\big)\\
&= \big(a_1\lambda^A_{b_1}(a_2)\lambda^A_{b_1\circ b_2\circ b_1^{\dagger}\circ b_2^{\dagger}}(\lambda_{b_2}(a_1)^{-1}a_2^{-1}),\, b_1\circ b_2\circ b_1^{\dagger}\circ b_2^{\dagger}\big).
\end{aligned}\tag{68}$$

Thus, we have

$$\widetilde{\xi_2}\,\theta_{\widetilde{A}}\big((\overline{a}_1,\overline{b}_1),\,(\overline{a}_2,\overline{b}_2)\big)$$

$$\begin{aligned}
&= \widetilde{\xi}_2\big(a_1\lambda^A_{b_1}(a_2)\lambda^A_{b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger}(\lambda_{b_2}(a_1)^{-1}a_2^{-1}), b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger\big), \quad \text{using (68)}\\
&= \big(\xi_2(a_1\lambda^A_{b_1}(a_2)\lambda^A_{b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger}(\lambda_{b_2}(a_1)^{-1}a_2^{-1})),\ \xi_2(b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger)\big). \qquad (69)
\end{aligned}$$

The isoclinism (ξ_1,ξ_2) from $(A,\cdot,\circ)$ onto $(B,\cdot,\circ)$ gives

$$\begin{aligned}
\xi_2(a\cdot b\cdot a^{-1}\cdot b^{-1}) &= \xi_1(a)\cdot\xi_1(b)\cdot\xi_1(a)^{-1}\cdot\xi_1(b)^{-1} \text{ and} \qquad (70)\\
\lambda^B_{\xi_1(a)}\big(\xi_1(b)\big)\cdot\big(\xi_1(b)\big)^{-1} &= \xi_2\big(\lambda^A_a(b)\cdot b^{-1}\big) \qquad (71)
\end{aligned}$$

for all $a,b\in A$. We compute the first coordinate of (69) and obtain

$$\begin{aligned}
&\xi_2(a_1\lambda^A_{b_1}(a_2)\lambda^A_{b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger}(\lambda_{b_2}(a_1)^{-1}a_2^{-1}))\\
&\quad= \xi_2(a_1\lambda^A_{b_1}(a_2)(\lambda^A_{b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger}(a_2\lambda^A_{b_2}(a_1)))^{-1})\\
&\quad= \xi_2(a_1\lambda^A_{b_1}(a_2)(a_2\lambda^A_{b_2}(a_1))^{-1}a_2\lambda^A_{b_2}(a_1)(\lambda^A_{b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger}(a_2\lambda^A_{b_2}(a_1)))^{-1})\\
&\quad= \xi_2\big(a_1\lambda^A_{b_1}(a_2)(a_2\lambda^A_{b_2}(a_1))^{-1}\big)\ \xi_2\big(a_2\lambda^A_{b_2}(a_1)(\lambda^A_{b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger}(a_2\lambda^A_{b_2}(a_1)))^{-1}\big)\\
&\quad= \xi_2\big(a_1\lambda^A_{b_1}(a_2)a_1^{-1}\lambda^A_{b_1}(a_2)^{-1}\lambda^A_{b_1}(a_2)a_1\lambda^A_{b_2^\dagger}(a_2\lambda^A_{b_2}(a_1))^{-1}\lambda^A_{b_2^\dagger}(a_2\lambda^A_{b_2}(a_1))(a_2\lambda^A_{b_2}(a_1))^{-1}\big)\\
&\qquad\quad \xi_1(a_2)\lambda^B_{\xi_1(b_2)}(\xi_1(a_1))(\lambda^B_{\xi_1(b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger)}(\xi_1(a_2)\lambda^B_{\xi_1(b_2)}(\xi_1(a_1))))^{-1},\ \text{using (71)}\\
&\quad= \xi_2\big(a_1\lambda^A_{b_1}(a_2)a_1^{-1}\lambda^A_{b_1}(a_2)^{-1}\big)\ \xi_2\big(\lambda^A_{b_1}(a_2)a_1\lambda^A_{b_2^\dagger}(a_2\lambda^A_{b_2}(a_1))^{-1}\big)\\
&\qquad\quad \xi_2\big(\lambda^A_{b_2^\dagger}(a_2\lambda^A_{b_2}(a_1))(a_2\lambda^A_{b_2}(a_1))^{-1}\big)\ \xi_1(a_2)\ \lambda^B_{\xi_1(b_2)}(\xi_1(a_1))\\
&\qquad\quad (\lambda^B_{\xi_1(b_1)\circ\xi_1(b_2)\circ\xi_1(b_1)^\dagger\circ\xi_1(b_2)^\dagger}(\xi_1(a_2)\lambda^B_{\xi_1(b_2)}(\xi_1(a_1))))^{-1},\ \text{by [21, Proposition 3.9]}\\
&\quad= \xi_1(a_1)\lambda^B_{\xi_1(b_1)}(\xi_1(a_2))\xi_1(a_1)^{-1}\lambda^B_{\xi_1(b_1)}(\xi_1(a_2))^{-1}\\
&\qquad\quad \xi_2\big(\lambda^A_{b_1}(a_2)a_2^{-1}a_2\lambda^A_{b_2^\dagger}(a_2)^{-1}\big)\lambda^B_{\xi_1(b_2)^\dagger}(\xi_1(a_2)\lambda^B_{\xi_1(b_2)}(\xi_1(a_1)))(\xi_1(a_2)\lambda^B_{\xi_1(b_2)}(\xi_1(a_1)))^{-1}\\
&\qquad\quad \xi_1(a_2)\lambda^B_{\xi_1(b_2)}(\xi_1(a_1))(\lambda^B_{\xi_1(b_1)\circ\xi_1(b_2)\circ\xi_1(b_1)^\dagger\circ\xi_1(b_2)^\dagger}(\xi_1(a_2)\lambda^B_{\xi_1(b_2)}(\xi_1(a_1))))^{-1},\\
&\qquad\quad \text{using (70) and (71)}\\
&\quad= \xi_1(a_1)\lambda^B_{\xi_1(b_1)}(\xi_1(a_2))\xi_1(a_1)^{-1}\lambda^B_{\xi_1(b_1)}(\xi_1(a_2))^{-1}\\
&\qquad\quad \lambda^B_{\xi_1(b_1)}(\xi_1(a_2))\,\xi_1(a_2)^{-1}\,\xi_1(a_2)\,\lambda^B_{\xi_1(b_2)^\dagger}(\xi_1(a_2))^{-1}\,\lambda^B_{\xi_1(b_2)^\dagger}(\xi_1(a_2)\lambda^B_{\xi_1(b_2)}(\xi_1(a_1)))\\
&\qquad\quad (\xi_1(a_2)\lambda^B_{\xi_1(b_2)}(\xi_1(a_1)))^{-1}\,\xi_1(a_2)\,\lambda^B_{\xi_1(b_2)}(\xi_1(a_1))\\
&\qquad\quad (\lambda^B_{\xi_1(b_1)\circ\xi_1(b_2)\circ\xi_1(b_1)^\dagger\circ\xi_1(b_2)^\dagger}(\xi_1(a_2)\lambda^B_{\xi_1(b_2)}(\xi_1(a_1))))^{-1}, \text{using (71)}\\
&\quad= \xi_1(a_1)\lambda^B_{\xi_1(b_1)}(\xi_1(a_2))\lambda^B_{\xi_1(b_1)\circ\xi_1(b_2)\circ\xi_1(b_1)^\dagger\circ\xi_1(b_2)^\dagger}(\lambda^B_{\xi_1(b_2)}(\xi_1(a_1))^{-1}\xi_1(a_2)^{-1}).
\end{aligned}$$

Using this in (69), we get

$$\begin{aligned}
&\widetilde{\xi}_2\,\theta_{\widetilde{A}}\big((\overline{a}_1,\overline{b}_1),\ (\overline{a}_2,\overline{b}_2)\big)\\
&= \big(\xi_2(a_1\lambda^A_{b_1}(a_2)\lambda^A_{b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger}(\lambda_{b_2}(a_1)^{-1}a_2^{-1})), \xi_2(b_1\circ b_2\circ b_1^\dagger\circ b_2^\dagger)\big)\\
&= \big(\xi_1(a_1)\lambda^B_{\xi_1(b_1)}(\xi_1(a_2))\lambda^B_{\xi_1(b_1)\circ\xi_1(b_2)\circ\xi_1(b_1)^\dagger\circ\xi_1(b_2)^\dagger}(\lambda^B_{\xi_1(b_2)}(\xi_1(a_1))^{-1}\xi_1(a_2)^{-1}),
\end{aligned}$$

$$
\begin{aligned}
& \xi_1(b_1) \circ \xi_1(b_2) \circ \xi_1(b_1)^{\dagger} \circ \xi_1(b_2)^{\dagger}\big), \quad \text{by [21, Proposition 3.9]} \\
= \;& (\xi_1(a_1), \xi_1(b_1)) \bullet (\xi_1(a_2), \xi_1(b_2)) \bullet (\xi_1(a_1), \xi_1(b_1))^{-1} \bullet (\xi_1(a_2), \xi_1(b_2))^{-1} \\
= \;& \theta_{\widetilde{B}}(\widetilde{\xi_1} \times \widetilde{\xi_1})\big((\overline{a}_1, \overline{b}_1), (\overline{a}_1, \overline{b}_1)\big), \quad \text{using Lemma (6.4)(2).}
\end{aligned}
$$

This shows that the left side of the diagram commutes. For the right hand diagram, observe that

$$
\begin{aligned}
\theta^*_{\widetilde{A}}\big((\overline{a}_1, \overline{b}_1), (\overline{a}_2, \overline{b}_2)\big) &= \big(1, b_1^{\dagger} \circ a_1\big) \bullet \big(a_2, b_2\big) \bullet \big(1, b_1^{\dagger} \circ a_1\big)^{-1} \bullet \big(a_2, b_2\big)^{-1} \\
&= \big(\lambda^A_{b_1^{\dagger} \circ a_1}(a_2), b_1^{\dagger} \circ a_1 \circ b_2\big) \bullet \big(\lambda^A_{a_1^{\dagger} \circ b_1}(\lambda^A_{b_2^{\dagger}}(a_2^{-1})), a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}\big) \\
&= \big(\lambda^A_{b_1^{\dagger} \circ a_1}(a_2) \cdot \lambda^A_{b_1^{\dagger} \circ a_1 \circ b_2}(\lambda^A_{a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}}(a_2^{-1})), b_1^{\dagger} \circ a_1 \circ b_2 \circ a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}\big) \\
&= \big(\lambda^A_{b_1^{\dagger} \circ a_1}(a_2) \cdot \lambda^A_{b_1^{\dagger} \circ a_1 \circ b_2 \circ a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}}(a_2^{-1}), b_1^{\dagger} \circ a_1 \circ b_2 \circ a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}\big). \qquad (72)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& \widetilde{\xi_2}\theta^*_{\widetilde{A}}\big((\overline{a}_1, \overline{b}_1),\, (\overline{a}_2, \overline{b}_2)\big) \\
&= \widetilde{\xi_2}\big(\lambda^A_{b_1^{\dagger} \circ a_1}(a_2) \cdot \lambda^A_{b_1^{\dagger} \circ a_1 \circ b_2 \circ a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}}(a_2^{-1}), b_1^{\dagger} \circ a_1 \circ b_2 \circ a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}\big), \text{ using (72)} \\
&= \big(\xi_2\big(\lambda^A_{b_1^{\dagger} \circ a_1}(a_2) \cdot \lambda^A_{b_1^{\dagger} \circ a_1 \circ b_2 \circ a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}}(a_2^{-1})\big),\ \xi_2\big(b_1^{\dagger} \circ a_1 \circ b_2 \circ a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}\big)\big) \\
&= \big(\xi_2\big(\lambda^A_{b_1^{\dagger} \circ a_1}(a_2) \cdot a_2^{-1} \cdot a_2 \cdot \lambda^A_{b_1^{\dagger} \circ a_1 \circ b_2 \circ a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}}(a_2^{-1})\big), \\
&\qquad \xi_1(b_1^{\dagger} \circ a_1) \circ \xi_1(b_2) \circ \xi_1(a_1^{\dagger} \circ b_1) \circ \xi_1(b_2)^{\dagger}\big), \text{ by [21, Proposition 3.9]} \\
&= \big(\xi_2\big(\lambda^A_{b_1^{\dagger} \circ a_1}(a_2) \cdot a_2^{-1} \cdot (\lambda^A_{b_1^{\dagger} \circ a_1 \circ b_2 \circ a_1^{\dagger} \circ b_1 \circ b_2^{\dagger}}(a_2) \cdot a_2^{-1})^{-1}\big), \\
&\qquad \xi_1(b_1^{\dagger} \circ a_1) \circ \xi_1(b_2) \circ \xi_1(a_1^{\dagger} \circ b_1) \circ \xi_1(b_2)^{\dagger}\big) \\
&= \big(\lambda^B_{\xi_1(b_1^{\dagger} \circ a_1)}(\xi_1(a_2)) \cdot \xi_1(a_2)^{-1} \cdot (\lambda^B_{\xi_1(b_1^{\dagger} \circ a_1) \circ \xi_1(b_2) \circ \xi_1(a_1^{\dagger} \circ b_1) \circ \xi_1(b_2)^{\dagger}}(\xi_1(a_2)) \cdot \xi_1(a_2)^{-1})^{-1}, \\
&\qquad \xi_1(b_1)^{\dagger} \circ \xi_1(a_1) \circ \xi_1(b_2) \circ \xi_1(a_1)^{\dagger} \circ \xi_1(b_1) \circ \xi_1(b_2)^{\dagger}\big), \text{ using (71)} \\
&= \big(\lambda^B_{\xi_1(b_1)^{\dagger} \circ \xi_1(a_1)}(\xi_1(a_2)) \cdot \lambda^B_{\xi_1(b_1)^{\dagger} \circ \xi_1(a_1) \circ \xi_1(b_2) \circ (\xi_1(b_1)^{\dagger} \circ \xi_1(a_1))^{\dagger} \circ \xi_1(b_2)^{\dagger}}(\xi_1(a_2)^{-1}), \\
&\qquad \xi_1(b_1)^{\dagger} \circ \xi_1(a_1) \circ \xi_1(b_2) \circ (\xi_1(b_1)^{\dagger} \circ \xi_1(a_1))^{\dagger} \circ \xi_1(b_2)^{\dagger}\big) \\
&= \big(1,\ \xi_1(b_1)^{\dagger} \circ \xi_1(a_1)\big) \bullet \big(\xi_1(a_2),\ \xi_1(b_2)\big) \bullet \big(1,\ \xi_1(b_1)^{\dagger} \circ \xi_1(a_1)\big)^{-1} \bullet \big(\xi_1(a_2),\ \xi_1(b_2)\big)^{-1} \\
&= \theta^*_{\widetilde{B}}\big(\big(\xi_1(\overline{a}_1), \xi_1(\overline{b}_1)\big), \big(\xi_1(\overline{a}_2), \xi_1(\overline{b}_2)\big)\big), \text{ using Lemma (6.4)(2)} \\
&= \theta^*_{\widetilde{B}}\,(\widetilde{\xi_1} \times \widetilde{\xi_1})\big((\overline{a}_1, \overline{b}_1), (\overline{a}_2, \overline{b}_2)\big).
\end{aligned}
$$

This completes the proof of the theorem. □

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Totally compatible structures on the radical of an incidence algebra

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Abstract. We describe totally compatible structures on the Jacobson radical of the incidence algebra of a finite poset over a field. We show that such structures are in general non-proper.

Contents

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Introduction

Two algebraic structures from a variety $\mathcal{A}$ defined on a same vector space are said to be *compatible* if their sum (equivalently, any linear combination) belongs to $\mathcal{A}$. Originated in the 70's [14], compatibility of Lie [8,9] associative [4,17] and Poisson [2,3,18] algebras has been actively studied in the context of mathematical physics. Recently, it has attracted the interest of scientists from other areas of mathematics who expanded the investigation of compatibility to the varieties of L_∞-algebras [5], Leibniz algebras [15], pre-Lie algebras [1], anti-pre-Lie algebras [16], left-symmetric algebras [23], Hom-Lie algebras [6], Hom-Lie triple systems [22] and so on. In particular, [1, 13, 15] provide algebraic classifications of low-dimensional compatible algebras.

Given a structure μ from $\mathcal{A}$, it is natural to ask: can one explicitly describe all the structures from $\mathcal{A}$ (on the same vector space) that are compatible with μ? This question has been previously studied [4, 17] for the variety of associative algebras. Recall that the compatibility of two *associative* bilinear products $\cdot_1$ and $\cdot_2$ on a vector space V is equivalent to the following identity on V:

$$(a \cdot_1 b) \cdot_2 c + (a \cdot_2 b) \cdot_1 c = a \cdot_1 (b \cdot_2 c) + a \cdot_2 (b \cdot_1 c). \tag{1}$$

In general, fixing $\cdot_1$, it is technically difficult to give a complete description of $\cdot_2$ satisfying (1), although there are some natural classes of such products associated to $\cdot_1$.

In this context, it seems to be more reasonable to study particular cases of (1) that are based on equalities of pairs of monomials. More precisely, let σ be a permutation of $\{1, 2\}$. If the i-th monomial of the left-hand side of (1) equals the $\sigma(i)$-th monomial of the right-hand side of (1) for all $i \in \{1, 2\}$, then $\cdot_1$ and $\cdot_2$ are said to be *σ-matching* [24] (in the case $\sigma = \mathrm{id}$ the triple $(V, \cdot_1, \cdot_2)$ is called a *matching dialgebra* [26], also known as *$As^{(2)}$-algebra* [28]). If all the 4 monomials in (1) are equal, then $\cdot_1$ and $\cdot_2$ are *totally compatible* (see [25, 27]). In [11], inspired by [21], we introduced another similar notion, called *interchangeability*, which means that the operations $\cdot_1$ and $\cdot_2$ can be permuted in each of the monomials in (1) without changing the brackets. If A is *unital* (or in certain sense close to being unital), then it is easy to see that σ-matching, interchangeable and totally compatible structures on A are simple modifications of the original associative product on A (see [11]).

The situation becomes more interesting in the context of *nilpotent* associative algebras, where new classes of compatible structures arise. In [12] we characterized σ-matching, interchangeable and totally compatible structures on the strictly upper triangular matrix algebra $UT_n(K)$, $n \geq 3$, which is a classical example of a nilpotent associative algebra. Notice that $UT_n(K)$ is the *Jacobson radical* of the algebra $T_n(K)$ of *all* upper triangular $n \times n$ matrices over K, which is in turn a particular case of the *incidence algebra* $I(X, K)$ of a *finite* poset X over K. So, our next goal was to generalize the results of [12] to the Jacobson radical $J(I(X, K))$ of $I(X, K)$. However, having classified the id-matching structures on $J(I(X, K))$ in the first draft of this manuscript, we gave up on the idea to proceed to (12)-matching and interchangeable structures, since the classification turned out to be too complex. We thus focused only on *totally compatible* structures that are the main object of study in this article.

In Section 1 we collect all the needed definitions and preliminary results about totally compatible structures and incidence algebras to make the paper self-contained.

In Section 2 we prove a couple of short general results whose particular cases will be used below.

In Section 3 we introduce *annihilator-valued* structures and study their relationship with structures $*_\varphi$ determined by centroid elements (known from [11]). As a result, we arrive at the definition of a *proper* totally compatible structure $*$, which essentially means that, up to an annihilator-valued component, $*$ is determined by a centroid element. The motivation comes from [11, 12], where most of the algebras had only totally compatible structures of this type. In Proposition 3.14, we prove that the class of proper totally compatible structures is closed under isomorphisms and antiisomorphisms.

Having prepared all the necessary general background, in Section 4 we focus on the algebra $J(I(X,K))$ and first describe in Proposition 4.7 its centroid, which is itself an interesting object of study. It is then used in Corollary 4.8 to give a description of proper totally compatible structures on $J(I(X,K))$.

Finally, in Section 5 we prove our main result Theorem 5.12, which gives a complete description of totally compatible structures on $J(I(X,K))$. Annihilator-valued structures again play an important role in this case, but, instead of $*_\varphi$, we obtain a more general class of structures that behave locally as $*_\varphi$. In Corollary 5.13, we find a sufficient condition on X under which all the totally compatible structures on $J(I(X,K))$ are proper. Example 5.15 shows that non-proper totally compatible structures on $J(I(X,K))$ may indeed exist, and Example 5.15 demonstrates that the condition from Corollary 5.13 is not necessary. We then give Example 5.17 that illustrates the situation when all the totally compatible structures on $J(I(X,K))$ are annihilator-valued. In fact, this happens if and only if X has length at most 2, as proved in Proposition 5.19. We leave as an open problem to describe finite posets X such that all the totally compatible structures on $J(I(X,K))$ are proper.

1 Definitions and preliminaries

Throughout the paper K will be an arbitrary field. All the algebras and vector spaces will be over K and all the products will be K-bilinear. As usual, the symbol $\circ$ will be reserved for the composition of two maps.

Let $(A,\cdot)$ be a (not necessarily associative) algebra. The *annihilator* of A is the ideal

$$\mathrm{Ann}(A,\cdot)=\{a\in A : a\cdot b=b\cdot a=0 \text{ for all } b\in A\}.$$

We will often write simply $\mathrm{Ann}(A)$, when it is clear what product is meant. Given subsets $B,C\subseteq A$, we denote

$$B\cdot C=\mathrm{span}_K\{b\cdot c : b\in B, c\in C\}.$$

We may write B^2 for $B \cdot B$, when the product on A is clear from the context. The *centroid* of A, denoted $\Gamma(A)$, is the space of linear maps $\varphi : A \to A$ such that

$$\varphi(a \cdot b) = a \cdot \varphi(b) = \varphi(a) \cdot b \tag{2}$$

for all $a, b \in A$.

1.1 Matching compatibilities and interchangeability

Generalizing [11, Definition 1.1] (inspired by [24]) we say that two (not necessarily associative!) products $\cdot_1$ and $\cdot_2$ on a same vector space V are

1. *σ-matching* (where $\sigma \in S_2 = \{\mathrm{id}, (12)\}$), if

$$(a \cdot_1 b) \cdot_2 c = a \cdot_{\sigma(1)} (b \cdot_{\sigma(2)} c) \text{ and } (a \cdot_2 b) \cdot_1 c = a \cdot_{\sigma(2)} (b \cdot_{\sigma(1)} c); \tag{3}$$

2. *interchangeable* if

$$(a \cdot_1 b) \cdot_2 c = (a \cdot_2 b) \cdot_1 c \text{ and } a \cdot_1 (b \cdot_2 c) = a \cdot_2 (b \cdot_1 c);$$

3. *totally compatible*, if

$$(a \cdot_1 b) \cdot_2 c = (a \cdot_2 b) \cdot_1 c = a \cdot_1 (b \cdot_2 c) = a \cdot_2 (b \cdot_1 c), \tag{4}$$

for all $a, b, c \in V$. Observe that σ-matching *associative* products are compatible in the sense of (1), while interchangeable ones need not be compatible in general. Furthermore, notice that $\cdot_1$ and $\cdot_2$ are totally compatible if and only if $\cdot_1$ and $\cdot_2$ are σ-matching *for all* $\sigma \in S_2$ if and only if $\cdot_1$ and $\cdot_2$ are interchangeable and σ-matching *for some* $\sigma \in S_2$.

Given an *associative* algebra $(A, \cdot)$, by a *σ-matching* (resp. *interchangeable* or *totally compatible*) *structure* on $(A, \cdot)$ we mean an *associative* product $*$ on A such that $\cdot$ and $*$ are σ-matching (resp. interchangeable or totally compatible). It is obvious that $\cdot$ is a totally compatible structure on $(A, \cdot)$. It is known that any *mutation* [7] $\cdot_x$ of $\cdot$ by $x \in A$, i.e. $a \cdot_x b = a \cdot x \cdot b$, is an id-matching structure on $(A, \cdot)$. If, moreover, x is central, then $\cdot_x$ is a totally compatible structure on $(A, \cdot)$.

As we saw in [11, Propositions 2.2 and 2.6], for a *unital* algebra A, any id-matching (resp. (12)-matching or interchangeable) structure on $(A, \cdot)$ is a mutation of $\cdot$ by $x \in A$ (resp. by a central $x \in A$). Consequently, any (12)-matching or interchangeable structure on a unital algebra $(A, \cdot)$ is totally compatible. If A is non-unital (in particular, if A is *nilpotent*), all these compatibilities may be different (see [11, Examples 1.4 and 1.5]).

1.2 Isomorphic and antiisomorphic structures

Let $*_1$ and $*_2$ be two σ-matching, interchangeable or totally compatible structures on an associative algebra $(A, \cdot)$. Following [11, 12], we say that an automorphism (resp.

antiautomorphism) ϕ of $(A,\cdot)$ is an *isomorphism* (resp. *antiisomorphism*) between $*_1$ and $*_2$, if

$$\phi(a *_1 b) = \phi(a) *_2 \phi(b) \text{ (resp. } \phi(a *_1 b) = \phi(b) *_2 \phi(a))$$

for all $a, b \in A$. The structures $*_1$ and $*_2$ are *isomorphic* (resp. *antiisomorphic*), if there is an isomorphism (resp. antiisomorphism) between them. This means that $(A,\cdot,*_1)$ and $(A,\cdot,*_2)$ are isomorphic (resp. antiisomorphic) as algebras with two multiplications. Given a σ-matching, interchangeable or totally compatible structure $*$ on $(A,\cdot)$ and an automorphism (resp. antiautomorphism) ϕ of $(A,\cdot)$, the product $\star$ on A defined by

$$a \star b = \phi(\phi^{-1}(a) * \phi^{-1}(b)) \text{ (resp. } a \star b = \phi(\phi^{-1}(b) * \phi^{-1}(a))) \tag{5}$$

for all $a, b \in A$, is a structure on $(A,\cdot)$ of the same kind as $*$, and it is isomorphic (resp. antiisomorphic) to $*$ (see [12, Lemma 1.2]).

1.3 Posets

Let $(X,\le)$ be a finite poset. As usual, we write $x < y$ to mean $x \le y$ and $x \ne y$. The binary relations $\ge$ and $>$ are inverse to $\le$ and $<$, respectively. Two elements $x, y \in X$ are said to be *comparable*, if $x \le y$ or $x \ge y$. A nonempty subset $C \subseteq X$ is a *chain*, if any two elements $x, y \in C$ are comparable. A chain $C \subseteq X$ is *maximal*, if there is no chain properly containing C. The *length* of a chain $C \subseteq X$ is $l(C) := |C| - 1$. The *length* of X is

$$l(X) := \max\{l(C) : C \text{ is a chain in } X\}.$$

Given $x \le y$, define

$$l(x, y) := l(\{z \in X : x \le z \le y\}).$$

Denote by $\min(X)$ and $\max(X)$ the subsets of minimal and maximal elements of X, respectively. Finally, write

$$X^n_< = \{(x_1, \dots, x_n) \in X^n : x_1 < \cdots < x_n\}.$$

1.4 Incidence algebras

Let $(X,\le)$ be a finite poset and K a field. The *incidence algebra* $I(X,K)$ of X over K (see [19, 20]) is the associative K-algebra with basis $\{e_{xy} : x, y \in X, x \le y\}$ (called *natural basis*) and bilinear multiplication

$$e_{xy} \cdot e_{uv} = \delta_{yu} e_{xv}$$

for all $x \le y$ and $u \le v$ in X. Here and below δ means the Kronecker delta. Given $f \in I(X,K)$, denote by $f(x,y)$ the coefficient of e_{xy} in the linear combination

$$f = \sum_{x \le y} f(x,y) e_{xy}.$$

The following formulas will be useful:

$$e_{xy}\cdot f=\sum_{v\ge y}f(y,v)e_{xv}\text{ and }f\cdot e_{xy}=\sum_{u\le x}f(u,x)e_{uy}.$$

Recall from [20, Theorem 4.2.5] that the *Jacobson radical* of $I(X,K)$ is

$$J(I(X,K))=\{f\in I(X,K):f(x,x)=0,\forall x\in X\}=\mathrm{span}_K\{e_{xy}:x<y\}.$$

The following facts are well-known and easy to prove:

$$\begin{aligned}\mathrm{Ann}(J(I(X,K)))&=\mathrm{span}_K\{e_{xy}:\min(X)\ni x<y\in\max(X)\},\\ J(I(X,K))\cdot J(I(X,K))&=\mathrm{span}_K\{e_{xy}:l(x,y)>1\}.\end{aligned}$$

To shorten some formulas, we will write $\mathcal{J}$ for $J(I(X,K))$ below.

2 Some general properties of totally compatible structures

The following is clear.

Lemma 2.1. *Let $\{\cdot_i\}_{i\in I}$ be a family of bilinear products on a same vector space V. If $\{\cdot_i\}_{i\in I}$ are pairwise σ-matching (resp., interchangeable or totally compatible), then any two finite linear combinations of $\{\cdot_i\}_{i\in I}$ are σ-matching (resp., interchangeable or totally compatible).*

The next result permits us to avoid unnecessary proofs of associativity in some cases.

Lemma 2.2. *Let $(A,\cdot)$ be an associative algebra and $*_1$, $*_2$ two bilinear products on A, where $*_1$ is associative. Assume that $\cdot$, $*_1$ and $*_2$ are pairwise σ-matching (resp. totally compatible). Then $*_1+*_2$ is a σ-matching (resp. totally compatible) structure on $(A,\cdot)$ if and only if $*_2$ is associative.*

Proof. It is enough to prove the result for σ-matching structures.

The "only if" part. Assume that $*_1+*_2$ is a σ-matching structure on $(A,\cdot)$. In particular, $*_1+*_2$ is associative. By Lemma 2.1 the products $*_1+*_2$ and $*_1$ are σ-matching, whence they are compatible. It follows that $*_2=(*_1+*_2)-*_1$ is associative.

The "if" part. Assume that $*_2$ is associative. Since $*_1$ and $*_2$ are σ-matching and associative, their sum $*_1+*_2$ is also associative. By Lemma 2.1 the products $*_1+*_2$ and $\cdot$ are σ-matching. Thus, $*_1+*_2$ is a σ-matching structure on $(A,\cdot)$. ☐

The following property is a particular case of total compatibility that will occur below.

Definition 2.3. Let $\cdot_1$ and $\cdot_2$ be two (not necessarily associative) products on a same vector space V. We say that $\cdot_1$ and $\cdot_2$ are *mutually annihilating* if

$$(V\cdot_1 V)\cdot_2 V=(V\cdot_2 V)\cdot_1 V=V\cdot_1(V\cdot_2 V)=V\cdot_2(V\cdot_1 V)=\{0\}.\tag{6}$$

3 Proper totally compatible structures

Recall from [11, Definition 2.4] the following.

Definition 3.1. Let $(A, \cdot)$ be a (not necessarily associative) algebra and $\varphi \in \Gamma(A)$. The product $*_\varphi$ on A given by

$$a *_\varphi b := \varphi(a \cdot b) \tag{7}$$

is said to be *determined* by φ.

Remark 3.2. In particular, $\cdot$ is determined by id.

The next fact was proved as a part of [11, Lemma 2.3].

Lemma 3.3. *Let $(A, \cdot)$ be an associative algebra and $\varphi \in \Gamma(A)$. Then $*_\varphi$ is a totally compatible structure on $(A, \cdot)$.*

Let us introduce one more class of structures, which plays a particularly important role in the case of nilpotent algebras.

Definition 3.4. Let $(A, \cdot)$ be a (not necessarily associative) algebra. A bilinear product $\bullet$ on A is said to be *annihilator-valued* (with respect to $\cdot$), if $\cdot$ and $\bullet$ are mutually annihilating. An *annihilator-valued structure* on an *associative* algebra $(A, \cdot)$ is an *associative* annihilator-valued bilinear product on A.

Remark 3.5. A bilinear product $\bullet$ on A is annihilator-valued if and only if

1. $A \bullet A \subseteq \mathrm{Ann}(A, \cdot)$;
2. $A \cdot A \subseteq \mathrm{Ann}(A, \bullet)$.

Remark 3.6. Any annihilator-valued product $\bullet$ on A is totally compatible with $\cdot$.

Remark 3.7. An annihilator-valued product $\bullet$ on A may be non-associative even if $(A, \cdot)$ is associative. For example, consider A to be an algebra with the trivial product, i.e. $A \cdot A = \{0\}$, so that $\mathrm{Ann}(A, \cdot) = A$. Then any bilinear product $\bullet$ on A is annihilator-valued.

Remark 3.8. If $\mathrm{Ann}(A, \cdot) \subseteq A \cdot A$, then any annihilator-valued product $\bullet$ on A satisfies $A \bullet A \subseteq \mathrm{Ann}(A, \bullet)$, so $\bullet$ is associative in this case, regardless of whether $(A, \cdot)$ is associative.

Annihilator-valued products on $(A, \cdot)$ admit the following constructive description.

Lemma 3.9. *Let $(A, \cdot)$ be a (not necessarily associative) algebra and V a vector space complement of $A \cdot A$ in A. Then a bilinear product $\bullet$ on A is annihilator-valued if and only if there exists a bilinear map $\mu : V \times V \to \mathrm{Ann}(A, \cdot)$ such that*

$$(a_1 + a_2) \bullet (b_1 + b_2) = \mu(a_2, b_2), \tag{8}$$

where $a_1, b_1 \in A \cdot A$ *and* $a_2, b_2 \in V$*.*

Proof. The "if" part. Let $\mu : V \times V \to \mathrm{Ann}(A, \cdot)$ be a bilinear map and define $\bullet$ by (8). It is clear that $\bullet$ is bilinear and $A \bullet A \subseteq \mathrm{Ann}(A, \cdot)$. Now, if $a \in A \cdot A$, then $a = a_1 + a_2$, where $a_1 = a \in A \cdot A$ and $a_2 = 0 \in V$. Hence, $a \bullet b = \mu(0, b_2) = 0$ for any $b = b_1 + b_2 \in A$ with $b_1 \in A \cdot A$ and $b_2 \in V$. Thus, $(A \cdot A) \bullet A = \{0\}$. The proof that $A \bullet (A \cdot A) = \{0\}$ is similar.

The "only if" part. Let $\bullet$ be an annihilator-valued product on A. Given $a = a_1 + a_2$ and $b = b_1 + b_2$ with $a_1, b_1 \in A \cdot A$ and $a_2, b_2 \in V$, by Remark 3.5(2) we have

$$(a_1 + a_2) \bullet (b_1 + b_2) = a_2 \bullet b_2.$$

So, we may define $\mu : V \times V \to \mathrm{Ann}(A, \cdot)$ by $\mu(u, v) = u \bullet v$ for all $u, v \in V$. Then μ is bilinear and (8) holds. ☐

Lemma 3.10. *Let* $(A, \cdot)$ *be a (not necessarily associative) algebra,* $\varphi \in \Gamma(A)$ *and* $\bullet$ *an annihilator-valued product on* A*. Then* $*_\varphi$ *and* $\bullet$ *are mutually annihilating, in particular, totally compatible.*

Proof. For all $a, b, c \in A$ we have

$$(a *_\varphi b) \bullet c = \varphi(a \cdot b) \bullet c = (a \cdot \varphi(b)) \bullet c = 0$$

by Remark 3.5(2) and

$$(a \bullet b) *_\varphi c = \varphi((a \bullet b) \cdot c) = \varphi(0) = 0$$

by Remark 3.5(1). Symmetrically, $a *_\varphi (b \bullet c) = a \bullet (b *_\varphi c) = 0$. ☐

Hence, as a consequence of Lemmas 3.3, 3.10, 2.2, and Remark 3.6, we get.

Corollary 3.11. *Let* $(A, \cdot)$ *be an associative algebra,* $\varphi \in \Gamma(A)$ *and* $\bullet$ *an annihilator-valued product on* A*. Then* $*_\varphi + \bullet$ *is a totally compatible structure on* $(A, \cdot)$ *if and only if* $\bullet$ *is associative, i.e.* $\bullet$ *is an annihilator-valued structure on* $(A, \cdot)$*.*

Definition 3.12. Let $(A, \cdot)$ be an associative algebra. A totally compatible structure on $(A, \cdot)$ is said to be *proper*, if it is of the form $*_\varphi + \bullet$, where $\varphi \in \Gamma(A)$ and $\bullet$ is an annihilator-valued structure on $(A, \cdot)$.

In most cases studied before in [11,12] all the totally compatible structures turned out to be proper.

Example 3.13. Let $(A, \cdot)$ be an associative algebra and $*$ a totally compatible structure on $(A, \cdot)$.

1. If A is unital, then $*$ is determined by $\varphi \in \Gamma(A) \cong C(A)$, where $C(A)$ is the center of A. Hence, $*$ is proper (see [11, Proposition 2.6]).

2. If A is an algebra with the trivial product, then $*$ is annihilator-valued, whence $*$ is proper (see Remark 3.7).
3. If A is the semigroup algebra of a rectangular band, then $*$ is determined by $\varphi \in \Gamma(A)$, whence $*$ is proper (see [11, Proposition 3.11]).
4. If A has enough idempotents, then $*$ is determined by $\varphi \in \Gamma(A)$, whence $*$ is proper (see [11, Proposition 3.23]).
5. If A is the free non-unital algebra over a set X with $|X| > 1$, then $*$ is determined by $\varphi \in \Gamma(A)$, whence $*$ is proper (see [11, Proposition 4.6]).
6. If A is the free non-unital commutative algebra over a set X with $|X| > 1$, then $*$ is determined by $\varphi \in \Gamma(A)$, whence $*$ is proper (see [11, Proposition 4.12]).
7. If A is the strictly upper triangular matrix algebra, then $*$ is proper (see [12, Theorem 5.1], where any linear combination of the structures $\mathbf{T}^1_{i,j}$ is annihilator-valued and any scalar multiple of the structure $\mathbf{T}^2$ is determined by $\varphi \in \Gamma(A)$).
8. If A is the free non-unital algebra over a set X with $|X| = 1$ (i.e., the non-unital polynomial algebra in one variable), then $*$ is not always proper (see [11, Remark 4.10]).

Proposition 3.14. *The class of proper totally compatible structures on an associative algebra* $(A, \cdot)$ *is closed under isomorphisms and antiisomorphisms.*

Proof. Let $* = *_\varphi + \bullet$, where $\varphi \in \Gamma(A)$ and $\bullet$ is an annihilator-valued structure on $(A, \cdot)$.

Any structure $\star$ isomorphic to $*$ is of the form $a \star b = \phi(\phi^{-1}(a) * \phi^{-1}(b))$ for some $\phi \in \mathrm{Aut}(A, \cdot)$. Then for all $a, b \in A$ we have

$$\begin{aligned} a \star b &= \phi(\phi^{-1}(a) * \phi^{-1}(b)) = \phi(\phi^{-1}(a) *_\varphi \phi^{-1}(b) + \phi^{-1}(a) \bullet \phi^{-1}(b)) \\ &= \phi(\varphi(\phi^{-1}(a) \cdot \phi^{-1}(b))) + \phi(\phi^{-1}(a) \bullet \phi^{-1}(b)) \\ &= (\phi \circ \varphi \circ \phi^{-1})(a \cdot b) + \phi(\phi^{-1}(a) \bullet \phi^{-1}(b)). \end{aligned}$$

Now observe that $\phi \circ \varphi \circ \phi^{-1} \in \Gamma(A)$, because

$$\begin{aligned} (\phi \circ \varphi \circ \phi^{-1})(a \cdot b) &= \phi(\varphi(\phi^{-1}(a) \cdot \phi^{-1}(b))) = \phi(\phi^{-1}(a) \cdot \varphi(\phi^{-1}(b))) \\ &= a \cdot \phi(\varphi(\phi^{-1}(b))) = a \cdot (\phi \circ \varphi \circ \phi^{-1})(b) \end{aligned}$$

and similarly $(\phi \circ \varphi \circ \phi^{-1})(a \cdot b) = (\phi \circ \varphi \circ \phi^{-1})(a) \cdot b$. Furthermore, the bilinear product

$$a \blacklozenge b = \phi(\phi^{-1}(a) \bullet \phi^{-1}(b))$$

on A is clearly associative, because $\bullet$ is, and satisfies $A \blacklozenge A \subseteq \mathrm{Ann}(A, \cdot)$ thanks to Remark 3.5(1) and $\phi(\mathrm{Ann}(A, \cdot)) = \mathrm{Ann}(A, \cdot)$. One also has $(A \cdot A) \blacklozenge A = A \blacklozenge (A \cdot A) = \{0\}$ due to Remark 3.5(2), $\phi^{-1}(A) = A$ and $\phi^{-1}(A \cdot A) = A \cdot A$. Thus, $\blacklozenge$ is annihilator-valued, so that $\star$ is proper as being equal to $*_{\phi \circ \varphi \circ \phi^{-1}} + \blacklozenge$.

If $\star$ is antiisomorphic to $*$, then it is given by $a \star b = \phi(\phi^{-1}(b) * \phi^{-1}(a))$ for some antiisomorphism ϕ of $(A, \cdot)$. Similarly to the isomorphic structure, we have

$$\begin{aligned} a \star b &= \phi(\varphi(\phi^{-1}(b) \cdot \phi^{-1}(a))) + \phi(\phi^{-1}(b) \bullet \phi^{-1}(a)) \\ &= (\phi \circ \varphi \circ \phi^{-1})(a \cdot b) + \phi(\phi^{-1}(b) \bullet \phi^{-1}(a)), \end{aligned}$$

where

$$\begin{aligned} (\phi \circ \varphi \circ \phi^{-1})(a \cdot b) &= \phi(\varphi(\phi^{-1}(b) \cdot \phi^{-1}(a))) = \phi(\varphi(\phi^{-1}(b)) \cdot \phi^{-1}(a)) \\ &= a \cdot \phi(\varphi(\phi^{-1}(b))) = a \cdot (\phi \circ \varphi \circ \phi^{-1})(b) \end{aligned}$$

and $(\phi \circ \varphi \circ \phi^{-1})(a \cdot b) = (\phi \circ \varphi \circ \phi^{-1})(a) \cdot b$, so that $\phi \circ \varphi \circ \phi^{-1} \in \Gamma(A)$. It is easily seen as above that $a \blacklozenge b = \phi(\phi^{-1}(b) \bullet \phi^{-1}(a))$ is an annihilator-valued structure on $(A, \cdot)$. Thus, $* = *_{\phi \circ \varphi \circ \phi^{-1}} + \blacklozenge$ is proper. $\square$

4 Proper totally compatible structures on $(\mathcal{J}, \cdot)$

4.1 The centroid of $\mathcal{J}$

In order to characterize proper totally compatible structures on $(\mathcal{J}, \cdot)$, we need a description of the centroid of $\mathcal{J}$. Let us introduce a class of centroid elements that will be one of the ingredients of the future description.

Definition 4.1. Let $(A, \cdot)$ be an algebra (which is not necessarily associative). A linear map $\varphi : A \to A$ is said to be *annihilator-valued*, if it satisfies

1. $\varphi(A) \subseteq \mathrm{Ann}(A)$;
2. $\varphi(A \cdot A) = \{0\}$.

Remark 4.2. Any annihilator-valued linear map $\varphi : A \to A$ belongs to $\Gamma(A)$, since all the products in (2) are zero.

Another class of centroid elements has its origin in [10]. Recall from [10] that a map $\sigma : X^2_< \to K$ is said to be *constant on chains* if

$$\sigma(x, y) = \sigma(u, v), \text{ whenever } x < y \text{ and } u < v \text{ belong to a same chain in } X.$$

Definition 4.3. Let $\sigma : X^2_< \to K$ be constant on chains. Define the linear map $\varphi_\sigma : \mathcal{J} \to \mathcal{J}$ by

$$\varphi_\sigma(e_{xy}) = \sigma(x, y)e_{xy},$$

for all $x < y$.

Lemma 4.4. *Let $\sigma : X^2_< \to K$ be constant on chains. Then $\varphi_\sigma \in \Gamma(\mathcal{J})$.*

Proof. Let $x < y$ and $u < v$. If $v \neq x$, then

$$e_{uv} \cdot \varphi_\sigma(e_{xy}) = \sigma(x,y)e_{uv} \cdot e_{xy} = 0 = \varphi_\sigma(0) = \varphi_\sigma(e_{uv} \cdot e_{xy}).$$

Otherwise, if $v = x$, then $x < y$ and $u < y$ belong to the same chain, $u < x < y$. Hence, $\sigma(x,y) = \sigma(u,y)$, so

$$e_{uv} \cdot \varphi_\sigma(e_{xy}) = \sigma(x,y)e_{uv} \cdot e_{xy} = \sigma(x,y)e_{uy} = \sigma(u,y)e_{uy} = \varphi_\sigma(e_{uy}) = \varphi_\sigma(e_{uv} \cdot e_{xy}).$$

Similarly, one proves that $\varphi_\sigma(e_{xy})e_{uv} = \varphi_\sigma(e_{xy} \cdot e_{uv})$. Thus, $\varphi_\sigma \in \Gamma(\mathcal{J})$. □

Lemma 4.5. *Let $\varphi \in \Gamma(\mathcal{J})$ and $x < y$.*

1. *If $l(x,y) > 1$, then $\varphi(e_{xy}) \in \operatorname{span}_K\{e_{xy}\}$.*
2. *If $l(x,y) = 1$, then $\varphi(e_{xy}) \in \operatorname{span}_K\{e_{xy}\} + \operatorname{Ann}(\mathcal{J})$.*

Proof. We first prove part (1). If $l(x,y) > 1$, then there is $x < z < y$, so

$$\varphi(e_{xy}) = \varphi(e_{xz} \cdot e_{zy}) = e_{xz} \cdot \varphi(e_{zy}) \in \operatorname{span}_K\{e_{xv} : z < v\}.$$

Similarly,

$$\varphi(e_{xy}) = \varphi(e_{xz}) \cdot e_{zy} \in \operatorname{span}_K\{e_{uy} : u < z\}.$$

Since

$$\operatorname{span}_K\{e_{xv} : z < v\} \cap \operatorname{span}_K\{e_{uy} : u < z\} = \operatorname{span}_K\{e_{xy}\},$$

the proof of part (1) is complete.

Now we prove part (2). Let $u < v$ with $(u,v) \neq (x,y)$. Assume first that $u \notin \min(X)$ and choose $a < u$. Then

$$\varphi(e_{xy})(u,v) = (e_{au} \cdot \varphi(e_{xy}))(a,v) = \varphi(e_{au} \cdot e_{xy})(a,v). \tag{9}$$

We have two cases.

Case 1: $u \neq x$. Then $e_{au} \cdot e_{xy} = 0$, so $\varphi(e_{xy})(u,v) = 0$ by (9).

Case 2: $u = x$ and $v \neq y$. Then $e_{au} \cdot e_{xy} = e_{ay}$, where $l(a,y) > 1$. By part (1) we have $\varphi(e_{ay}) \in \operatorname{span}_K\{e_{ay}\}$. Since $v \neq y$, then $\varphi(e_{ay})(a,v) = 0$, so $\varphi(e_{xy})(u,v) = 0$ in view of (9). One similarly shows that $\varphi(e_{xy})(u,v) = 0$, whenever $v \notin \max(X)$. This completes the proof of part (2). □

Lemma 4.6. *Let $\varphi \in \Gamma(\mathcal{J})$. Then the associated map $\sigma : X^2_< \to K$, given by*

$$\sigma(x,y) = \varphi(e_{xy})(x,y),$$

for all $x < y$, is constant on chains.

Proof. Let $x < y$ and C be a chain containing x and y. Denote the least element of C by a and the greatest element of C by b. We first show that $\sigma(x, y) = \sigma(a, y)$. It suffices to consider the case $a \neq x$. Then $a < x$, so

$$\varphi(e_{xy})(x, y) = (e_{ax} \cdot \varphi(e_{xy}))(a, y) = \varphi(e_{ax} \cdot e_{xy})(a, y) = \varphi(e_{ay})(a, y),$$

whence $\sigma(x, y) = \sigma(a, y)$. Similarly, one proves that $\sigma(a, y) = \sigma(a, b)$. This implies that $\sigma(x, y) = \sigma(a, b)$. □

Proposition 4.7. *The elements of* $\Gamma(\mathcal{J})$ *are exactly the maps of the form* $\varphi_\sigma + \eta$*, where* $\sigma : X^2_< \to K$ *is constant on chains and* η *is annihilator-valued.*

Proof. Let $\varphi \in \Gamma(\mathcal{J})$. Then there is $\sigma : X^2_< \to K$ from Lemma 4.6 which is constant on chains. Define $\eta = \varphi - \varphi_\sigma$. In view of Lemma 4.5 we have $\eta(e_{xy}) \in \mathrm{Ann}(\mathcal{J})$ and $\eta(e_{xy}) = 0$, whenever $l(x, y) > 1$. The latter means that $\eta(\mathcal{J} \cdot \mathcal{J}) = \{0\}$. Thus, η is annihilator-valued.

Conversely, $\varphi_\sigma \in \Gamma(\mathcal{J})$ by Lemma 4.4 and $\eta \in \Gamma(\mathcal{J})$ by Remark 4.2. Therefore, $\varphi_\sigma + \eta \in \Gamma(\mathcal{J})$. □

4.2 The description of proper totally compatible structures

Corollary 4.8. *Proper totally compatible structures on* $(\mathcal{J}, \cdot)$ *are exactly bilinear products* $*$ *of the form*

$$e_{xy} * e_{uv} = \sigma(x, v)\delta_{yu}e_{xv} + e_{xy} \bullet e_{uv}, \tag{10}$$

where $\sigma : X^2_< \to K$ *is constant on chains and* $\bullet$ *is an annihilator-valued structure on* $(\mathcal{J}, \cdot)$*.*

Proof. This follows from Definition 3.12 and Proposition 4.7, because we have

$$e_{xy} *_\varphi e_{uv} = \varphi_\sigma(e_{xy} \cdot e_{uv}) + \eta(e_{xy} \cdot e_{uv}) = \varphi_\sigma(\delta_{yu}e_{xv}) = \sigma(x, v)\delta_{yu}e_{xv},$$

for $\varphi = \varphi_\sigma + \eta \in \Gamma(\mathcal{J})$, as in Proposition 4.7. □

Remark 4.9. Let $\sim$ be the equivalence relation on $X^2_<$ generated by the pairs

$$((x, y), (u, v)) \in X^2_< \times X^2_<,$$

such that there exists a chain in X containing both $x < y$ and $u < v$. Then a map $X^2_< \to K$ is constant on chains if and only if it is constant on $\sim$-classes of $X^2_<$.

Taking a $\sim$-class $\mathcal{C}$ of $X^2_<$, one can define $\sigma_{\mathcal{C}} : X^2_< \to K$ by

$$\sigma_{\mathcal{C}}(x, y) = \begin{cases} 1, & (x, y) \in \mathcal{C}, \\ 0, & (x, y) \notin \mathcal{C}. \end{cases}$$

Then $\sigma_{\mathcal{C}}$ is constant on chains, and the corresponding totally compatible structure $*_{\mathcal{C}}$ on $(\mathcal{J}, \cdot)$ determined by $\varphi_{\sigma_{\mathcal{C}}} \in \Gamma(\mathcal{J})$ is given by

$$e_{xy} *_{\mathcal{C}} e_{uv} = \begin{cases} e_{xv}, & y = u \text{ and } (x, v) \in \mathcal{C}, \\ 0, & \text{otherwise.} \end{cases}$$

It is therefore easily seen by Corollary 4.8 that the totally compatible structures on $(\mathcal{J}, \cdot)$ are exactly bilinear products $*$ that are sums of an annihilator-valued structure $\bullet$ on $(\mathcal{J}, \cdot)$ and a linear combination of the structures $*_{\mathcal{C}}$, where $\mathcal{C}$ runs through the set $X^2_</{\sim}$ of $\sim$-classes of $X^2_<$.

5 Totally compatible structures on $(\mathcal{J}, \cdot)$

5.1 A class of totally compatible structures

We first define a combinatorial notion that will play a role in the main result.

Definition 5.1. Let $(x, y, z), (u, v, w) \in X^3_<$. The triples (x, y, z) and (u, v, w) are said to be *chained* if there is a chain in X containing both $x < y < z$ and $u < v < w$. Let $\approx$ be the equivalence relation generated by all the pairs of chained triples in $X^3_<$ and denote by $X^3_</{\approx}$ the set of $\approx$-equivalence classes of $X^3_<$.

Remark 5.2. If (x, y, z) and (u, v, w) are chained, then $(x, z) \sim (u, w)$. Consequently, for any $(x, y, z), (u, v, w) \in X^3_<$ we have

$$(x, y, z) \approx (u, v, w) \Rightarrow (x, z) \sim (u, w). \tag{11}$$

The converse of (11) is not true in general (see Examples 5.15, 5.16, 5.17 below).

Lemma 5.3. *Let $\mathcal{C} \in X^3_</{\approx}$. Then the bilinear product $*_{\mathcal{C}}$ given by*

$$e_{xy} *_{\mathcal{C}} e_{uv} = \begin{cases} e_{xv}, & y = u \text{ and } (x, y, v) \in \mathcal{C}, \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

is a totally compatible structure on $(\mathcal{J}, \cdot)$.

Proof. Let $x < y$, $z < u$ and $v < w$. We are going to show that

$$(e_{xy} \cdot e_{zu}) *_{\mathcal{C}} e_{vw} = (e_{xy} *_{\mathcal{C}} e_{zu}) \cdot e_{vw} = e_{xy} \cdot (e_{zu} *_{\mathcal{C}} e_{vw}) = e_{xy} *_{\mathcal{C}} (e_{zu} \cdot e_{vw}). \tag{13}$$

Case 1: $y \neq z$. Then $e_{xy} \cdot e_{zu} = e_{xy} *_{\mathcal{C}} e_{zu} = 0$, so the first two products of (13) are zero. By (12) the product $e_{zu} *_{\mathcal{C}} e_{vw}$ is either e_{zw} or 0. In any case, $e_{xy} \cdot (e_{zu} *_{\mathcal{C}} e_{vw}) = 0$. Similarly, $e_{zu} \cdot e_{vw}$ is either e_{zw} or 0. In any case, $e_{xy} *_{\mathcal{C}} (e_{zu} \cdot e_{vw}) = 0$ by (12).

Case 2: $y = z$. Then $(e_{xy} \cdot e_{zu}) *_{\mathcal{C}} e_{vw} = e_{xu} *_{\mathcal{C}} e_{vw}$. We have the following two subcases.

Case 2.1: $u \neq v$. Then $e_{xu} *_{\mathcal{C}} e_{vw} = e_{zu} \cdot e_{vw} = e_{zu} *_{\mathcal{C}} e_{vw} = 0$, so all the products of (13), except possibly for the second one, are zero. Since $e_{xy} *_{\mathcal{C}} e_{zu}$ is either e_{xu} or 0, we have $(e_{xy} *_{\mathcal{C}} e_{zu}) \cdot e_{vw} = 0$ as well.

Case 2.2: $u = v$. Observe that

$$(x, y, u) \approx (x, u, w) \approx (y, u, w) \approx (x, y, w), \tag{14}$$

because $x < y < u < w$. We have the following two subcases.

Case 2.2.1: $(x, u, w) \notin \mathcal{C}$. Then $(x, y, u), (y, u, w), (x, y, w) \notin \mathcal{C}$. By (12) we have $e_{xu} *_{\mathcal{C}} e_{vw} = 0$, so that $(e_{xy} \cdot e_{zu}) *_{\mathcal{C}} e_{vw} = 0$. Since $e_{xy} *_{\mathcal{C}} e_{zu} = e_{zu} *_{\mathcal{C}} e_{vw} = 0$, then $(e_{xy} *_{\mathcal{C}} e_{zu}) \cdot e_{vw} = e_{xy} \cdot (e_{zu} *_{\mathcal{C}} e_{vw}) = 0$. Furthermore, $e_{xy} *_{\mathcal{C}} (e_{zu} \cdot e_{vw}) = e_{xy} *_{\mathcal{C}} e_{zw}$, which is also zero by (12).

Case 2.2.2: $(x, u, w) \in \mathcal{C}$. Then $(x, y, u), (y, u, w), (x, y, w) \in \mathcal{C}$. In this case, we have $e_{xu} *_{\mathcal{C}} e_{vw} = e_{xw}$, so that $(e_{xy} \cdot e_{zu}) *_{\mathcal{C}} e_{vw} = e_{xw}$ by (12). Now, by (12):

$$\begin{aligned}
(e_{xy} *_{\mathcal{C}} e_{zu}) \cdot e_{vw} &= e_{xu} \cdot e_{vw} = e_{xw},\\
e_{xy} \cdot (e_{zu} *_{\mathcal{C}} e_{vw}) &= e_{xy} \cdot e_{zw} = e_{xw},\\
e_{xy} *_{\mathcal{C}} (e_{zu} \cdot e_{vw}) &= e_{xy} *_{\mathcal{C}} e_{zw} = e_{xw}.
\end{aligned}$$

Thus, the proof of (13) is complete.

For the associativity of $*$ let us show that

$$(e_{xy} *_{\mathcal{C}} e_{zu}) *_{\mathcal{C}} e_{vw} = e_{xy} *_{\mathcal{C}} (e_{zu} *_{\mathcal{C}} e_{vw}). \tag{15}$$

The proof of (15) is similar to that of (13). As above, one sees that both of the monomials of (15) are zero, whenever $y \neq z$ or $u \neq v$. If $y = z$ and $u = v$, then $x < y < u < w$, so that (14) holds. Hence, both of the monomials of (15) are either zero (if $(x, y, u) \notin \mathcal{C}$) or are equal to e_{xw} (if $(x, y, u) \in \mathcal{C}$). □

Lemma 5.4. *Let $\mathcal{C}$ be a $\approx$-class and $\bullet$ an annihilator-valued product on $\mathcal{J}$. Then $*_{\mathcal{C}}$ and $\bullet$ are mutually annihilating, in particular, totally compatible.*

Proof. It follows from $\mathcal{J} \bullet \mathcal{J} \subseteq \mathrm{Ann}(\mathcal{J}, \cdot)$ and (12) that $(\mathcal{J} \bullet \mathcal{J}) *_{\mathcal{C}} \mathcal{J} = \mathcal{J} *_{\mathcal{C}} (\mathcal{J} \bullet \mathcal{J}) = \{0\}$. Now, since $\mathcal{J} *_{\mathcal{C}} \mathcal{J} \subseteq \mathcal{J} \cdot \mathcal{J}$ and $\mathcal{J} \cdot \mathcal{J} \subseteq \mathrm{Ann}(\mathcal{J}, \bullet)$, then $\mathcal{J} *_{\mathcal{C}} \mathcal{J} \subseteq \mathrm{Ann}(\mathcal{J}, \bullet)$. □

Lemma 5.5. *For any pair of distinct $\approx$-classes $\mathcal{C}$ and $\mathcal{D}$, the structures $*_{\mathcal{C}}$ and $*_{\mathcal{D}}$ are mutually annihilating, in particular, totally compatible.*

Proof. Let $x < y$, $z < u$ and $v < w$.

Case 1: $y \neq z$ or $u \neq v$. Then

$$\begin{aligned}
(e_{xy} *_{\mathcal{C}} e_{zu}) *_{\mathcal{D}} e_{vw} = (e_{xy} *_{\mathcal{D}} e_{zu}) *_{\mathcal{C}} e_{vw} &= e_{xy} *_{\mathcal{C}} (e_{zu} *_{\mathcal{D}} e_{vw})\\
&= e_{xy} *_{\mathcal{D}} (e_{zu} *_{\mathcal{C}} e_{vw}) = 0
\end{aligned} \tag{16}$$

by (12).

Case 2: $y = z$ and $u = v$. Then $x < y < u < w$, so we have (14).

Case 2.1: $(x, y, u) \in \mathcal{C}$. Then $(x, y, u), (x, u, w), (y, u, w), (x, y, w) \notin \mathcal{D}$ by (14), whence (16).

Case 2.2: $(x, y, u) \in \mathcal{D}$. Then $(x, y, u), (x, u, w), (x, y, w), (y, u, w) \notin \mathcal{C}$ by (14), whence (16).

Case 2.3: $(x, y, u) \notin \mathcal{C} \sqcup \mathcal{D}$. Then (16) is immediate in view of (14). □

Proposition 5.6. *The sum of an annihilator-valued structure* $\bullet$ *on* $(\mathcal{J}, \cdot)$ *and a linear combination of the structures* $*_\mathcal{C}$*, where* $\mathcal{C}$ *runs through the set of* $\approx$*-equivalence classes of* $X^3_<$*, is a totally compatible structure on* $(\mathcal{J}, \cdot)$*.*

Proof. By Lemma 2.1, Remark 3.6 and Lemmas 5.5, 5.4, 5.3, any product of the form $\bullet + \sum_{\mathcal{C}\in X^3_</\approx} \alpha_\mathcal{C} *_\mathcal{C}$ is totally compatible with $\cdot$. Since $\bullet$ and all the structures $*_\mathcal{C}$ are associative and pairwise totally compatible, then $\bullet + \sum_{\mathcal{C}\in X^3_</\approx} \alpha_\mathcal{C} *_\mathcal{C}$ is also associative. □

5.2 The description of totally compatible structures

Recall the following fact.

Lemma 5.7 (Lemma 4.1 from [12])**.** *Let* $(A, \cdot)$ *be a (not necessarily associative) algebra and* $*$ *a bilinear product on* A *such that* $*$ *and* $\cdot$ *are interchangeable. For all* $a, b \in A$ *if* $a \cdot b = 0$*, then* $a * b \in \mathrm{Ann}(A, \cdot)$*.*

Lemma 5.8. *Let* $*$ *be a bilinear product on* $\mathcal{J}$ *such that* $*$ *and* $\cdot$ *are* (12)*-matching. Given* $x < y$ *and* $u < v$ *with* $y \neq u$*, if* $l(x, y) > 1$ *or* $l(u, v) > 1$*, then* $e_{xy} * e_{uv} = 0$*.*

Proof. Assume that $l(x, y) > 1$ and choose $x < z < y$. Then

$$e_{xy} * e_{uv} = (e_{xz} \cdot e_{zy}) * e_{uv} = e_{xz} * (e_{zy} \cdot e_{uv}) = 0.$$

The case $l(u, v) > 1$ is similar. □

Lemma 5.9. *Assume that* $*$ *is a bilinear product on* $\mathcal{J}$ *such that* $*$ *and* $\cdot$ *are totally compatible. Let* $x < y < z$*.*

1. *If* $l(x, y) > 1$ *or* $l(y, z) > 1$*, then* $e_{xy} * e_{yz} \in \mathrm{span}_K\{e_{xz}\}$*.*
2. *If* $l(x, y) = l(y, z) = 1$*, then* $e_{xy} * e_{yz} \in \mathrm{span}_K\{e_{xz}\} + \mathrm{Ann}(\mathcal{J})$*.*

Proof. We first prove part (1). Assume that $l(x, y) > 1$ and choose $x < u < y$. Then

$$e_{xy} * e_{yz} = (e_{xu} \cdot e_{uy}) * e_{yz} = e_{xu} \cdot (e_{uy} * e_{yz}) \in \mathrm{span}_K\{e_{xv} : u < v\}.$$

On the other hand,

$$e_{xy} * e_{yz} = (e_{xu} \cdot e_{uy}) * e_{yz} = (e_{xu} * e_{uy}) \cdot e_{yz} \in \mathrm{span}_K\{e_{wz} : w < y\}.$$

Consequently,

$$e_{xy} * e_{yz} \in \operatorname{span}_K\{e_{xv} : u < v\} \cap \operatorname{span}_K\{e_{wz} : w < y\} = \operatorname{span}_K\{e_{xz}\}.$$

The case $l(y, z) > 1$ is similar.

Now we prove part (2). Let $l(x, y) = l(y, z) = 1$. Take $u < v$ such that $(u, v) \neq (x, z)$. Assume first that $u \notin \min(X)$ and choose $a < u$. Then

$$(e_{xy} * e_{yz})(u, v) = (e_{au} \cdot (e_{xy} * e_{yz}))(a, v).$$

We have two cases.

Case 1: $u \neq x$. Then $e_{au} \cdot (e_{xy} * e_{yz}) = (e_{au} \cdot e_{xy}) * e_{yz} = 0$.

Case 2: $u = x$ and $v \neq z$. By part (1) we have

$$e_{au} \cdot (e_{xy} * e_{yz}) = e_{ax} \cdot (e_{xy} * e_{yz}) = e_{ax} * (e_{xy} \cdot e_{yz}) = e_{ax} * e_{xz} \in \operatorname{span}_K\{e_{az}\},$$

because $l(x, z) > 1$. Since $v \neq z$, we conclude that $(e_{au} \cdot (e_{xy} * e_{yz}))(a, v) = 0$.

Thus, $(e_{xy} * e_{yz})(u, v) = 0$, whenever $(u, v) \neq (x, z)$ and $u \notin \min(X)$. One similarly proves that $(e_{xy} * e_{yz})(u, v) = 0$, whenever $(u, v) \neq (x, z)$ and $v \notin \max(X)$. □

Lemma 5.10. *Assume that $*$ is a bilinear product on $\mathcal{J}$ such that $*$ and $\cdot$ are totally compatible. Given $x < y < z$ and $u < v < w$, if $(x, y, z) \approx (u, v, w)$, then*

$$(e_{xy} * e_{yz})(x, z) = (e_{uv} * e_{vw})(u, w).$$

Proof. It is enough to consider the case where $x < y < z$ and $u < v < w$ are chained, so let C be a chain containing $x < y < z$ and $u < v < w$. Denote the least element of C by a and the greatest element of C by b. If $x \neq a$, then $a < x$ and

$$\begin{aligned}(e_{xy} * e_{yz})(x, z) = (e_{ax} \cdot (e_{xy} * e_{yz}))(a, z) &= ((e_{ax} \cdot e_{xy}) * e_{yz})(a, z) \\ &= (e_{ay} * e_{yz})(a, z). \end{aligned} \tag{17}$$

If $x = a$, then (17) trivially holds. Similarly,

$$(e_{ay} * e_{yz})(a, z) = (e_{ay} * e_{yb})(a, b). \tag{18}$$

Combining (17) and (18), we get $(e_{xy} * e_{yz})(x, z) = (e_{ay} * e_{yb})(a, b)$. For the same reason $(e_{uv} * e_{vw})(u, w) = (e_{av} * e_{vb})(a, b)$. If $y = v$, we are done. If $y < v$, then

$$e_{av} * e_{vb} = (e_{ay} \cdot e_{yv}) * e_{vb} = e_{ay} * (e_{yv} \cdot e_{vb}) = e_{ay} * e_{yb}.$$

The case $v < y$ is symmetric. □

Proposition 5.11. *Assume that $*$ is a totally compatible structure on $(\mathcal{J}, \cdot)$. Then $*$ is the sum of an annihilator-valued structure $\bullet$ on $(\mathcal{J}, \cdot)$ and a linear combination of the structures $*_{\mathcal{C}}$, where $\mathcal{C}$ runs through the set of $\approx$-equivalence classes of $X^3_<$.*

Proof. For all $x < y < z$ denote $\alpha_{xyz} = (e_{xy} * e_{yz})(x, z) \in K$. Furthermore, for all $x < y$ and $u < v$ with $l(x, y) = l(u, v) = 1$ denote $a^{uv}_{xy} = e_{xy} * e_{uv} - \alpha_{xyv}\delta_{yu}e_{xv}$. By Lemmas 5.7, 5.8 and 5.9, we have $a^{uv}_{xy} \in \mathrm{Ann}(\mathcal{J})$ and

$$e_{xy} * e_{uv} = \begin{cases} \alpha_{xyv}\delta_{yu}e_{xv} + a^{uv}_{xy}, & l(x, y) + l(u, v) = 2, \\ \alpha_{xyv}\delta_{yu}e_{xv}, & l(x, y) + l(u, v) > 2. \end{cases} \tag{19}$$

Moreover, by Lemma 5.10 we have $\alpha_{xyz} = \alpha_{uvw}$, whenever $(x, y, z) \approx (u, v, w)$.

Given a $\approx$-class $\mathcal{C}$, denote by $\alpha_{\mathcal{C}}$ the common α_{xyz} for all $(x, y, z) \in \mathcal{C}$. Furthermore, denote by $\bullet$ the following bilinear product on $\mathcal{J}$:

$$e_{xy} \bullet e_{uv} = \begin{cases} a^{uv}_{xy}, & l(x, y) + l(u, v) = 2, \\ 0, & l(x, y) + l(u, v) > 2. \end{cases}$$

Then $\bullet$ is annihilator-valued, and by (19) we have $* = \bullet + \sum_{\mathcal{C}\in X^3_</\approx} \alpha_{\mathcal{C}} *_{\mathcal{C}}$. According to Lemmas 5.3 and 5.5, the linear combination $\sum_{\mathcal{C}\in X^3_</\approx} \alpha_{\mathcal{C}} *_{\mathcal{C}}$ is a totally compatible structure on $(\mathcal{J}, \cdot)$. Furthermore, by Lemmas 5.4, 2.1 and 5.5, the products $\bullet$ and $\sum_{\mathcal{C}\in X^3_</\approx} \alpha_{\mathcal{C}} *_{\mathcal{C}}$ are totally compatible. Finally, $\bullet$ and $\cdot$ are totally compatible by Remark 3.6. Then it follows from Lemma 2.2 that $\bullet$ is associative, so that it is an annihilator-valued structure on $(\mathcal{J}, \cdot)$. □

Theorem 5.12. *Let X be a finite poset and K a field. Then totally compatible structures on $(\mathcal{J}, \cdot)$ are exactly sums of annihilator-valued structures $\bullet$ on $(\mathcal{J}, \cdot)$ and linear combinations of the structures $*_{\mathcal{C}}$, where $\mathcal{C}$ runs through the set of $\approx$-equivalence classes of $X^3_<$.*

Proof. It follows from Propositions 5.6 and 5.11. □

As a consequence, we obtain the following sufficient condition for all the totally compatible structures on $(\mathcal{J}, \cdot)$ to be proper.

Corollary 5.13. *Assume that for all $(x, y, z), (u, v, w) \in X^3_<$ one has*

$$(x, z) \sim (u, w) \Rightarrow (x, y, z) \approx (u, v, w). \tag{20}$$

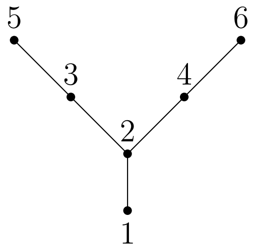
Then all the totally compatible structures on $(\mathcal{J}, \cdot)$ are proper.

Proof. Let $*$ be a totally compatible structure on $(\mathcal{J}, \cdot)$. Then $* = \bullet + \sum_{\mathcal{C}\in X^3_</\approx} \alpha_{\mathcal{C}} *_{\mathcal{C}}$ as in Theorem 5.12. Given $x < y$ with $l(x, y) > 1$, choose an arbitrary $x < z < y$ and define $\sigma(x, y) = \alpha_{\mathcal{C}}$, where $\mathcal{C}$ is the $\approx$-class of (x, z, y). By (20) the definition does not depend on the choice of z. Moreover, $\sigma(x, y) = \sigma(u, v)$, whenever $l(x, y) > 1$, $l(u, v) > 1$ and $(x, y) \sim (u, v)$ by (20). Now if $l(x, y) = 1$ and there is $(u, v) \sim (x, y)$ with $l(u, v) > 1$, then set $\sigma(x, y) := \sigma(u, v)$. This is again well-defined by (20). Finally, if $l(x, y) = 1$ and there is no $(u,v) \sim (x,y)$ with $l(u,v) > 1$, then the $\sim$-class of (x,y) is a singleton[1], so we can define $\sigma(x,y)$ arbitrarily. By construction, $\sigma : X^2_< \to K$ is constant on chains and (10) holds. Thus, $*$ is proper by Remark 4.9. $\square$

Remark 5.14. The condition (20) is not necessary for all the totally compatible structures on $(\mathcal{J},\cdot)$ to be proper as will be seen in Examples 5.16 and 5.17.

We first give an example showing that there may exist non-proper totally compatible structures on $(\mathcal{J},\cdot)$.

Example 5.15. Let $X = \{1,2,3,4,5,6\}$ with the partial order whose Hasse diagram is given below.



Observe that

$$X^3_< = \{(1,2,3),(1,2,4),(1,2,5),(1,2,6),(2,3,5),(2,4,6)\}.$$

The set $X^3_<$ decomposes into two $\approx$-classes:

$$\mathcal{C} = \{(1,2,3),(1,2,5),(2,3,5)\} \text{ and } \mathcal{D} = \{(1,2,4),(1,2,6),(2,4,6)\},$$

because all the elements constituting the triples from $\mathcal{C}$ belong to the chain $1 < 2 < 3 < 5$, all the elements constituting the triples from $\mathcal{D}$ belong to the chain $1 < 2 < 4 < 6$ and no triple from $\mathcal{C}$ is chained with a triple from $\mathcal{D}$. Thus, by Theorem 5.12 we obtain a totally compatible structure $*_{\mathcal{C}}$ on $(\mathcal{J},\cdot)$, which is given by

$$e_{12} *_{\mathcal{C}} e_{23} = e_{13},\ e_{12} *_{\mathcal{C}} e_{25} = e_{15},\ e_{23} *_{\mathcal{C}} e_{35} = e_{25}, \tag{21}$$

where the remaining products of basis elements are zero. However, $*_{\mathcal{C}}$ is not proper. For, if it were proper, by Corollary 4.8 there would exist a map $\sigma : X^2_< \to K$, constant on chains, and an annihilator-valued structure $\bullet$ on $(\mathcal{J},\cdot)$ such that (10) holds. Then we would have

$$e_{12} *_{\mathcal{C}} e_{23} = \sigma(1,3)e_{13} + e_{12} \bullet e_{23} \text{ and } e_{12} *_{\mathcal{C}} e_{24} = \sigma(1,4)e_{14} + e_{12} \bullet e_{24}. \tag{22}$$

Since

$$\operatorname{Ann}(\mathcal{J},\cdot) = \operatorname{span}_K\{e_{15}, e_{16}\},$$

---

[1] This happens exactly when $\{x,y\}$ is a maximal chain in X.

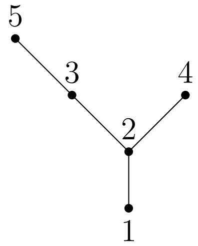
then comparing $e_{12} *_{\mathcal{C}} e_{23}$ in (21) and (22), we would conclude that $\sigma(1,3) = 1$. Similarly, from $e_{12} *_{\mathcal{C}} e_{24} = 0$ and (22) we would get $\sigma(1,4) = 0$. But σ is constant on chains, so taking the chains $1 < 2 < 3$ and $1 < 2 < 4$, we would have

$$\sigma(1,3) = \sigma(1,2) = \sigma(1,4),$$

a contradiction.

A slight modification of Example 5.15 results in only proper totally compatible structures on $(\mathcal{J}, \cdot)$.

Example 5.16. Let $X = \{1,2,3,4,5\}$ with the partial order whose Hasse diagram is given below.



Similarly to Example 5.15, we have

$$X^3_< = \{(1,2,3),(1,2,4),(1,2,5),(2,3,5)\}$$

with two $\approx$-classes

$$\mathcal{C} = \{(1,2,3),(1,2,5),(2,3,5)\} \text{ and } \mathcal{D} = \{(1,2,4)\}.$$

Let $*$ be a totally compatible structure on $(\mathcal{J}, \cdot)$. By Theorem 5.12 there are $\alpha, \beta \in K$ and an annihilator-valued structure $\bullet$ on $(\mathcal{J}, \cdot)$ such that

$$\begin{aligned}
&e_{12} * e_{23} = \alpha e_{13} + e_{12} \bullet e_{23},\ \ e_{12} * e_{25} = \alpha e_{15},\\
&e_{23} * e_{35} = \alpha e_{25} + e_{23} \bullet e_{35},\ \ e_{12} * e_{24} = \beta e_{14} + e_{12} \bullet e_{24},\\
&e_{xy} * e_{uv} = e_{xy} \bullet e_{uv},\ \text{if } y \neq u \text{ and } (x,y),(u,v) \in \{(1,2),(2,3),(2,4),(3,5)\},
\end{aligned}$$

where the remaining products of basis elements are zero. Define $\sigma : X^2_< \to K$ by setting $\sigma(x,y) = \alpha$ for all $(x,y) \in X^2_<$. Obviously, σ is constant on chains. Furthermore, define a bilinear product $\blacklozenge$ on $\mathcal{J}$ by setting, for all $x < y$ and $u < v$,

$$e_{xy} \blacklozenge e_{uv} = \begin{cases} e_{xy} \bullet e_{uv} + (\beta - \alpha)e_{14}, & (x,y) = (1,2) \text{ and } (u,v) = (2,4),\\ e_{xy} \bullet e_{uv}, & \text{otherwise.} \end{cases} \tag{23}$$

Since

$$\mathrm{Ann}(\mathcal{J}, \cdot) = \mathrm{span}_K\{e_{15}, e_{14}\},$$

then by (23) and Remark 3.5(1) we have

$$\mathcal{J} \blacklozenge \mathcal{J} \subseteq \mathcal{J} \bullet \mathcal{J} + \mathrm{Ann}(\mathcal{J}, \cdot) \subseteq \mathrm{Ann}(\mathcal{J}, \cdot).$$

Moreover, as

$$\mathcal{J} \cdot \mathcal{J} = \mathrm{span}_K\{e_{13}, e_{14}, e_{15}, e_{25}\},$$

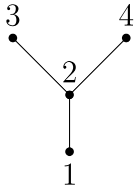
by (23) and Remark 3.5(2) we have

$$(\mathcal{J} \cdot \mathcal{J}) \blacklozenge \mathcal{J} = (\mathcal{J} \cdot \mathcal{J}) \bullet \mathcal{J} = \{0\} \text{ and } \mathcal{J} \blacklozenge (\mathcal{J} \cdot \mathcal{J}) = \mathcal{J} \bullet (\mathcal{J} \cdot \mathcal{J}) = \{0\}.$$

Finally, $\mathrm{Ann}(\mathcal{J}, \cdot) \subseteq \mathcal{J} \cdot \mathcal{J}$ guarantees that $\blacklozenge$ is associative by Remark 3.8. So, $\blacklozenge$ is an annihilator-valued structure on $(\mathcal{J}, \cdot)$. Thus, $*$ is proper by Corollary 4.8 (where $\bullet$ should be replaced by $\blacklozenge$).

An easier example, where all the totally compatible structures on $(\mathcal{J}, \cdot)$ are proper (in fact, annihilator-valued), is as follows.

Example 5.17. Let $X = \{1, 2, 3, 4\}$ with the partial order whose Hasse diagram is given below.



Then $X^3_< = \{(1, 2, 3), (1, 2, 4)\}$ having two $\approx$-classes $\mathcal{C} = \{(1, 2, 3)\}$ and $\mathcal{D} = \{(1, 2, 4)\}$. Let $*$ be a totally compatible structure on $(\mathcal{J}, \cdot)$. By Theorem 5.12 there are $\alpha, \beta \in K$ and an annihilator-valued structure $\bullet$ on $(\mathcal{J}, \cdot)$ such that

$$\begin{aligned} &e_{12} * e_{23} = \alpha e_{13} + e_{12} \bullet e_{23}, \ e_{12} * e_{24} = \beta e_{14} + e_{12} \bullet e_{24}, \\ &e_{xy} * e_{uv} = e_{xy} \bullet e_{uv}, \text{ if } y \neq u \text{ and } (x, y), (u, v) \in \{(1, 2), (2, 3), (2, 4)\}, \end{aligned}$$

where the remaining products of basis elements are zero. But

$$\mathrm{Ann}(\mathcal{J}, \cdot) = \mathrm{span}_K\{e_{13}, e_{14}\} = \mathcal{J} \cdot \mathcal{J},$$

so that $*$ is itself an annihilator-valued structure on $(\mathcal{J}, \cdot)$. In particular, $*$ is proper.

Remark 5.18. Observe that in Examples 5.17 and 5.16 all the pairs $(x, y) \in X^2_<$ are $\sim$-equivalent, but not all the triples $(x, y, z) \in X^3_<$ are $\approx$-equivalent. So, (20) does not hold.

In fact, the result of Example 5.17 can be generalized.

Proposition 5.19. *Let X be a finite poset. Then all the totally compatible structures on $(\mathcal{J},\cdot)$ are annihilator-valued if and only if $l(X)\le 2$.*

Proof. The "if" part. Assume that $l(X)\le 2$ and let $*$ be a totally compatible structure on $(\mathcal{J},\cdot)$. By Theorem 5.12 we have $*=\bullet+\sum_{\mathcal{C}\in X^3_</\approx}\alpha_{\mathcal{C}}*_{\mathcal{C}}$, where $\bullet$ is an annihilator-valued structure on $(\mathcal{J},\cdot)$.

We first prove that $\mathcal{J}*\mathcal{J}\subseteq \mathrm{Ann}(\mathcal{J},\cdot)$. Let $x<y$ and $u<v$. If $y\ne u$, then $e_{xy}*e_{uv}=e_{xy}\bullet e_{uv}\in\mathrm{Ann}(\mathcal{J},\cdot)$. Otherwise, $e_{xy}*e_{uv}=\alpha_{\mathcal{C}}e_{xv}+e_{xy}\bullet e_{uv}$, where $\mathcal{C}$ is the $\approx$-class containing (x,y,v). Observe that $x<y<v$ is a maximal chain (otherwise there would exist a maximal chain in X properly containing $x<y<v$, i.e. having length >2). Therefore, $x\in\min(X)$ and $v\in\max(X)$. This means that $e_{xv}\in\mathrm{Ann}(\mathcal{J},\cdot)$, so that $e_{xy}*e_{uv}\in\mathrm{Ann}(\mathcal{J},\cdot)$. Thus, $\mathcal{J}*\mathcal{J}\subseteq\mathrm{Ann}(\mathcal{J},\cdot)$.

Now let us show that $\mathcal{J}\cdot\mathcal{J}\subseteq\mathrm{Ann}(\mathcal{J},*)$. Let $x<y$ and $u<v$. Assume first that $l(x,y)>1$, i.e. $e_{xy}\in\mathcal{J}\cdot\mathcal{J}$. Then $y\ne u$, since otherwise $l(x,v)>2$ contradicting $l(X)=2$. Hence, $e_{xy}*e_{uv}=e_{xy}\bullet e_{uv}$, which is 0 by Remark 3.5(2). This proves $(\mathcal{J}\cdot\mathcal{J})*\mathcal{J}=\{0\}$. The proof of $\mathcal{J}*(\mathcal{J}\cdot\mathcal{J})=\{0\}$ is similar (it corresponds to the case $l(u,v)>1$). Thus, $*$ is annihilator-valued.

The "only if" part. Assume that $l(X)>2$. Then there are $x<y<z$ in X such that $x\notin\min(X)$ or $z\notin\max(X)$. The product $\cdot$ is itself a totally compatible structure on $(\mathcal{J},\cdot)$, and it is not annihilator-valued, because $e_{xy}\cdot e_{yz}=e_{xz}\notin\mathrm{Ann}(\mathcal{J},\cdot)$. □

6 Open problem

Problem 6.1. Let K be a field. Characterize finite posets X such that all the totally compatible structures on $(\mathcal{J},\cdot)$ are proper.

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Graph quandles: Generalized Cayley graphs of racks and right quasigroups

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Abstract. This article lays the foundations for an analogue of geometric group theory that studies actions on graphs by right quasigroups, including racks and quandles. We study markings of graphs that realize racks, and we introduce (di)graph invariants based on such markings. We show that all right quasigroups are realizable by edgeless graphs and complete (di)graphs. Using Schreier (di)graphs, we also characterize Cayley (di)graphs of right quasigroups Q that realize Q. In particular, all racks are realizable by their full Cayley (di)graphs. This solves two problems of Valeriy Bardakov. Finally, we give graph-theoretic characterizations of labeled Cayley digraphs of right-cancellative magmas, right-divisible magmas, right quasigroups, racks, quandles, involutory racks, and kei.

Contents

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1 Introduction

1.1 Motivating discussion

In 1992, Fenn and Rourke [10] introduced *racks* to develop complete invariants of framed links. Racks are generalizations of *quandles*, which Joyce [17] and Matveev [21] independently introduced in 1982 to develop complete invariants of unframed links. In turn, quandles generalize algebraic structures called *kei*, which Takasaki [30] introduced in 1942 to study Riemannian symmetric spaces. Racks and quandles play important roles in knot theory, low-dimensional topology, and quantum algebra, making it important to understand their structure. For general references on the theory, see [9, 23].

Similarly to groups, there is a rich literature on finite racks and quandles in algebra and topology (see [7] for references), but studying infinite racks is more difficult. In recent years, this problem has created interest in developing a more geometric understanding of racks and quandles [1]. To that end, various authors have adapted methods from geometric group theory to study racks, including Cayley graphs [16], certain metrics [16, 20], and even a bounded cohomology theory for racks and quandles [18, 20, 24].

Racks, quandles, and groups are special classes of nonassociative algebraic structures called *right quasigroups*, which are used to study column Latin squares [12], smooth deformations of Lie group structures [32], and nonassociative generalizations of Hopf algebras [25]. To develop an analogue of geometric group theory for racks and quandles, it is therefore natural to study actions on geometric objects by right quasigroups rather than groups. In this paper, we take those geometric objects to be graphs, particularly labeled Cayley digraphs and other (di)graphs obtained from racks.

1.1.1 Racks as symmetries

In 2020, Bardakov [1] introduced a way to construct right quasigroups from *markings* of graphs by graph automorphisms. Seeking geometric interpretations of racks, Bardakov posed the following questions about which marked graphs realize racks and how they relate to Cayley graphs of racks.

Problem 1.1. Under what conditions does a marked graph realize a rack or a quandle?

Problem 1.2. Given a rack or quandle Q, is there always a marked graph (Γ, R) that realizes Q? If so, can we choose Γ to be a Cayley graph of Q?

We answer Bardakov's questions with the following results. In the following, we refer to directed graphs as *digraphs* and simple undirected graphs as *graphs*.

Theorem 1.3 (see Theorem 5.1)**.** *Let* Γ *be a* (*di*)*graph, denote its vertex set by* V*, and let* $R : V \to \operatorname{Aut}\Gamma$ *be a marking* (*resp. q-marking*) *of* V*. Then the right quasigroup* V_R^Γ *realized by* Γ *is a rack* (*resp. quandle*) *if and only if* R *is a magma homomorphism from* V_R^Γ *to* $\operatorname{Conj}(\operatorname{Aut}\Gamma)$*.*

Proposition 1.4 (see Proposition 3.9)**.** *All right quasigroups are realizable by edgeless graphs and complete* (*di*)*graphs.*

Theorem 1.5 (see Theorems 6.1 and 6.2)**.** *Let* V_R *be a right quasigroup, and let* Γ *be a Cayley digraph* (*resp. graph*) *of* V_R *with connection set* $S \subseteq V$*. Then* (Γ, R) *realizes* V_R *if and only if, for all* $h, v \in V$ *and* $s \in S$*, there exists an element* $t \in S$ *such that*

$$R_t R_h(v) = R_h R_s(v) \quad (\textit{resp. } R_t^{\pm 1} R_v(w) = R_v R_s(w)).$$

Corollary 1.6 (see Corollary 6.4)**.** *All racks are realizable by their full Cayley* (*di*)*graphs.*

Theorem 5.1 answers Problem 1.1. Since all racks are right quasigroups, Proposition 3.9 answers a generalized version of the first question in Problem 1.2. Corollary 6.4 answers the second question in Problem 1.2, while Theorems 6.1 and 6.2 answer generalized forms of the question. As an application of marked graphs, we also introduce two integer invariants $\mu_{\text{rack}}, \mu_{\text{qnd}}$ of (di)graphs.

1.1.2 Characterization of labeled Cayley digraphs

Although this paper is the first to study labeled Cayley (di)graphs of right quasigroups in general, various authors have studied Cayley graphs of various classes of right quasigroups appearing in combinatorics, algebraic topology, and knot theory. For example, full Cayley digraphs of unital, fixed point-free right quasigroups have various applications in network theory [5], and full Cayley graphs of right quasigroups have applications in categorical covering theory [11].

Introduced by Winker [31] in 1984, full Cayley graphs of racks help classify finite quotients of fundamental quandles of links [6, 15] and generalizations of these quotients [22]. Full Cayley graphs of racks can even be interpreted as 1-skeletons of CW complexes called *extended rack spaces* and used to construct homotopy invariants of links [33]. A very recent application of Cayley graphs of racks makes it possible to study infinite quandles via methods adapted from geometric group theory [16].

In this light, a graph-theoretic rather than purely algebraic characterization of Cayley graphs of right quasigroups and racks is desirable. A question of Hamkins [14] calls for such a characterization for Cayley graphs of groups; in response, Caucal [2, 3] generalized this question to the settings of magmas and *labeled digraphs* or *labeled transition systems*, that is, digraphs with an assignment of directed edges (rather than vertices) to elements of a distinguished *labeling set*. Unlike Caucal, we assume the labeling set to be a subset of the vertex set.

Problem 1.7. Given a full subcategory $\mathcal{C}$ of magmas (e.g., groups, monoids, quandles), are there graph-theoretic conditions that characterize labeled Cayley digraphs of objects in $\mathcal{C}$?

Caucal answered Problem 1.7 for left quasigroups, quasigroups [2], semigroups, and various classes of monoids, including groups [3]. Note that the answers to Problem 1.7 for left and right quasigroups are distinct because the right-multiplication maps R_v of left quasigroups are not necessarily permutations. Chishwashwa et al. addressed a similar question for vertex-labeled Cayley digraphs of unital, fixed point-free right quasigroups [5]. In this paper, we answer Problem 1.7 for various classes of racks and right quasigroups.

Theorem 1.8 (see Theorems 7.12 and 7.14)**.** *Let $\mathcal{Q}$ be the class of all labeled digraphs that are deterministic, source-complete, codeterministic, and target-complete. Then $\mathcal{Q}$ is precisely the class of labeled Cayley digraphs of right quasigroups.*

Moreover, there exist graph theoretic-conditions on $\mathcal{Q}$ that restrict it to the subclasses of labeled Cayley digraphs of racks and quandles.

1.2 Structure of the paper

In Section 2, we discuss right quasigroups, racks, and quandles. In Section 3, we discuss (di)graphs, marked graphs in the sense of Bardakov, Cayley (di)graphs of magmas in the sense of Caucal, and Schreier graphs of group actions. In Section 4, we give examples of Cayley (di)graphs of right quasigroups and their markings. In Section 5, we answer Problem 1.1. As an application, we introduce two rack-theoretic (di)graph invariants $\mu_{\mathrm{rack}}, \mu_{\mathrm{qnd}}$ with general results for path graphs and cycle graphs. In Section 6, we answer Problem 1.2 and its analogues for right quasigroups and digraphs. In Section 7, we define labeled Cayley digraphs of magmas in the sense of Caucal, and we answer Problem 1.7 for right-cancellative magmas, right-divisible magmas, right quasigroups, racks, quandles, involutory racks, and kei. In Section 8, we propose directions for future research.

Notation

Given a positive integer $n \in \mathbb{Z}^+$, let $[n]$ denote the set $\{1, 2, \dots, n\}$. Denote the symmetric group of $[n]$ by S_n with its elements written in cycle notation, and denote the symmetric group of any other set X by S_X. We also denote the composition of functions $\varphi : V \to W$ and $\psi : W \to X$ by $\psi\varphi$, and we denote the identity map on a set V by id_V. For $n \geq 3$, let D_n be the dihedral group of order $2n$. Given a subset S of a group G, let $S^{-1} := \{s^{-1} \mid s \in S\}$.

2 Algebraic preliminaries

We recall the definitions of right quasigroups, racks, and quandles.

2.1 Magmas

Right quasigroups are examples of more general algebraic structures called *magmas*.

Definition 2.1. A *magma* or *groupoid* is a pair (V, R), denoted by V_R, where V is a set and R is a mapping from V to the set of functions from V to V. For all $v \in V$, we call the map $R_v := R(v)$ a *right-multiplication map* or *right-translation map*.

Moreover, we say that V_R is a *right quasigroup* if $R(V) \subseteq S_V$, that is, if all right-multiplication maps are permutations of V. We say that R is a *right quasigroup structure* on V.

Remark 2.2. Although the notation V_R is nonstandard, we use it because Bardakov [1] formulated his open problems with it.

Example 2.3. The (right) regular action $R : G \to S_G$ of a group G, given by $R_h(g) := gh$, is a right quasigroup structure on G. Thus, right quasigroups generalize groups.

Example 2.4. If we take $G := \mathbb{Q}$ to be the set of rational numbers in Example 2.3, then the positive rational numbers $\mathbb{Q}^+$ are a submagma of $\mathbb{Q}$ but not a right quasigroup.

Definition 2.5. Let V_R and W_T be magmas. A *magma homomorphism* from V_R to W_T is a function $\varphi : V \to W$ satisfying

$$\varphi R_v = T_{\varphi(v)}\varphi \quad \text{for all } v \in V.$$

Remark 2.6. Some authors define magmas as sets V equipped with some binary operation $\triangleleft : V \times V \to V$; homomorphisms are then defined in the obvious way. For right quasigroups, racks, and quandles, the operation $\triangleleft$ satisfies additional axioms like the right cancellation property.

This convention is equivalent to our convention via the formula

$$v \triangleleft w = R_w(v).$$

Indeed, it is easy to see that our notion of homomorphism is equivalent to the one using binary operations $\triangleleft$. Our convention appears in [7, 19]; we adopt this convention because it adapts more easily to graph-theoretic settings.

Evidently, magmas and right quasigroups form categories. In particular, we can consider *automorphism groups* $\operatorname{Aut} V_R$ of magmas.

2.2 Racks

Racks and quandles form important full subcategories of the category of right quasigroups.

Definition 2.7. Let V_R be a right quasigroup.

- The *right-multiplication group* $\operatorname{RMlt} V_R$ is the subgroup of S_V generated by all right-multiplication maps.
- We say that V_R is a *rack* if every right-multiplication map R_v is a magma endomorphism. Concretely, this means that

  $$R_v R_w = R_{R_v(w)} R_v \tag{1}$$

  for all $v, w \in V$. In this case, we call R a *rack structure* on V.
- Separately, we say that V_R is *involutory* if every right-multiplication map is an involution, that is, if $R_v^2 = \mathrm{id}_V$ for all $v \in V$.

Remark 2.8. A right quasigroup V_R is a rack if and only if $\operatorname{RMlt} V_R$ is a (normal) subgroup of $\operatorname{Aut} V_R$. In this case, some authors call $\operatorname{RMlt} V_R$ the *inner automorphism group* of V_R and denote it by $\operatorname{Inn} V_R$. Other authors denote $\operatorname{RMlt} V_R$ by $\operatorname{Mlt}_r V_R$.

Definition 2.9. Let V_R be a rack.

- We say that V_R is a *quandle* if $R_v(v) = v$ for all $v \in V$. In this case, we call R a *quandle structure* on V.
- If V_R is an involutory quandle, we call it a *kei*.

2.2.1 Examples

We discuss some common examples of right quasigroups, racks, and quandles. See Section 4 for further examples.

Example 2.10 ([23, Example 2.13])**.** Let G be a union of conjugacy classes in a group, and define $C : G \to S_G$ by sending any element $g \in G$ to the conjugation map C_g defined by

$$C_g(h) := ghg^{-1}.$$

Then $\operatorname{Conj} G := G_C$ is a quandle called a *conjugation quandle* or *conjugacy quandle*.

If G is a group, then $\operatorname{Conj} G$ is a kei if and only if $g^2 \in Z(G)$ for all $g \in G$. In particular, not all quandles are involutory.

Example 2.11 ([9, Example 99])**.** Let V be a set, fix a permutation $\sigma \in S_V$, and define $R_v := \sigma$ for all $v \in V$. Then the assignment R is a rack structure on V, and we call $V_\sigma := V_R$ a *permutation rack* or *constant action rack*.

Note that V_σ is a quandle if and only if $\sigma = \mathrm{id}_V$, in which case we call V_{id_V} a *trivial quandle*. In particular, not all racks are quandles. Moreover, V_σ is involutory if and only if σ is an involution.

Example 2.12. The (right) regular action of a group G is a rack structure if and only if G is the trivial group. In particular, not all right quasigroups are racks.

2.2.2 Preliminary results

An alternative characterization of racks does the heavy lifting in solving Problem 1.1; see Theorem 5.1.

Proposition 2.13. *Let V_R be a magma. Then V_R is a rack if and only if R is a magma homomorphism from V_R to* $\operatorname{Conj} S_V$.

Proof. To prove necessity, suppose that V_R is a rack. Then $R(V) \subseteq S_V$, and Equation (1) states that

$$RR_v(w) = R_{R_v(w)} = R_v R_w R_v^{-1} = C_{R_v}(R_w) = C_{R(v)}R(w)$$

for all $v, w \in V$. Hence, R is a magma homomorphism from V_R to $\operatorname{Conj} S_V$.

To prove sufficiency, suppose that R is a magma homomorphism from V_R to $\operatorname{Conj} S_V$. Similarly to before,

$$R_{R_v(w)} = RR_v(w) = C_{R(v)}R(w) = C_{R_v}(R_w) = R_v R_w R_v^{-1}$$

for all $v, w \in V$. Since $R_v \in S_V$ is bijective, we obtain Equation (1). □

In Sections 4 and 6, we employ a necessary condition for a right quasigroup to be a rack.

Lemma 2.14. *If V_R is a rack, then $R(V)$ is closed under conjugation.*

Proof. Equation (1) states that $R_t R_h(v) = R_h R_s(v)$ for all $v, w \in V$. □

3 Graph-theoretic preliminaries

We discuss directed and undirected graphs, marked graphs as constructed by Bardakov [1] and Cayley graphs of magmas as introduced by Caucal [2]. We also relate the latter to *Schreier graphs* of group actions. Since we only discuss labeled digraphs in Section 7, we defer defining them until then.

3.1 Graphs and digraphs

We recall the graph-theoretic constructions of Bardakov from [1]. Like Bardakov, we assume that all undirected graphs are simple, and we do not allow for digraphs to have multiple edges. However, we do allow for digraphs to have loops.

Definition 3.1.

- A *digraph* or *directed graph* Γ is a pair (V, E) where V is a set and E is a subset of $V \times V$. We say that V is the *vertex set* of Γ, and we say that E is the (*directed*) *edge set* or *arc set* of Γ, respectively.

- *Simple undirected graphs*, which we only call *graphs*, are defined similarly to digraphs, except that every element of E is an unordered pair of vertices $\{v, w\} \subseteq V$ such that $v \neq w$.

- Given a (di)graph Γ, we denote the vertex and edge sets of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. We say that $|V(\Gamma)|$ is the *order* of Γ.

Definition 3.2. Let $\Gamma = (V, E)$ be a digraph. An *automorphism* of Γ is a permutation $\varphi \in S_V$ such that $(\varphi(v), \varphi(w)) \in E$ for all edges $(v, w) \in E$. Automorphisms of graphs are defined similarly.

As with racks and quandles, digraphs and graphs form categories. In particular, we can consider automorphism groups $\operatorname{Aut}\Gamma$ of digraphs and graphs.

3.2 Marked graphs

We consider (di)graphs with *markings* and *q-markings* of their vertices as introduced by Bardakov [1].

Definition 3.3. Let Γ be a (di)graph with vertex set V.

- A *marking* of Γ is a function $R : V \to \operatorname{Aut}\Gamma$ with each image notated as $R_v := R(v)$. We say that the right quasigroup $V_R^\Gamma := (V, R)$ is *realized* by the *marked (di)graph* (Γ, R).

- Let R be a marking of Γ. We say that R is also a *q-marking* of Γ if $R_v(v) = v$ for all $v \in V$. In this case, we call (Γ, R) a *q-marked graph*.

- Conversely, we say that a right quasigroup Q is *realizable* by Γ if there exists a marking R of Γ such that $Q \cong V_R^\Gamma$.

Remark 3.4. Bardakov calls V_R^Γ a *graph groupoid*. We eschew this name to emphasize the fact that V_R^Γ is not only a magma but also a right quasigroup.

If V_R^Γ is also a rack (resp. quandle), Bardakov calls it a *graph rack* (resp. *graph quandle*).

Remark 3.5. If (Γ, R) is a q-marked graph, then V_R^Γ is a rack if and only if V_R^Γ is a quandle.

Remark 3.6. Since markings of a graph Γ are simply functions $R : V(\Gamma) \to \operatorname{Aut}\Gamma$, the number of right quasigroup structures on $V(\Gamma)$ whose right-multiplication maps lie in $\operatorname{Aut}\Gamma$ equals

$$|\operatorname{Aut}\Gamma|^{|V(\Gamma)|}.$$

Example 3.7. Given any (di)graph Γ, the trivial marking $v \mapsto \mathrm{id}_{V(\Gamma)}$ realizes a trivial quandle.

Example 3.8. Let $n \in \mathbb{Z}^+$ be a positive integer, and let Γ be the *star graph* $K_{1,n}$ of order $n+1$; cf. Table 1 in Section 5. Then $\operatorname{Aut}\Gamma \cong S_n$ acts on $V := V(\Gamma)$ by permuting the leaves.

If $n = 2$, it is easy to see that of the eight possible markings $R : V \to S_n$, exactly four are rack structures. Indeed, if $\ell_1, \ell_2 \in V$ are the two leaves and $v \in V$ is the central vertex, then Equation (1) forces $R_{\ell_1} = R_{\ell_2}$. For an example of a marking of $K_{1,2}$ that does *not* realize a rack, see Example 4.1.

We give more substantial examples following our discussion of Cayley digraphs; see Section 4.

3.2.1 Right quasigroups are realizable

We answer the first question in Problem 1.2. Recall that a digraph $\Gamma = (V, E)$ is called *complete* if

$$E = \{(v, w) \in V \times V \mid v \neq w\}.$$

Complete graphs are defined similarly.

Proposition 3.9. *All right quasigroups are realizable by edgeless graphs and complete (di)graphs.*

Proof. Given a right quasigroup V_R, let Γ be an edgeless or complete (di)graph with vertex set V. Then $\operatorname{Aut}\Gamma = S_V$, so $R : V \to S_V$ is a marking of Γ. Hence, $V_R^\Gamma = V_R$. ☐

Remark 3.10. In Bardakov's original wording [1], Proposition 3.9 implies that every groupoid (resp. rack, quandle) is a graph groupoid (resp. graph rack, graph quandle).

3.3 Cayley graphs

Having established Proposition 3.9, it is natural to ask which racks are realized by (di)graphs with more intricate structures. To that end, we discuss Cayley (di)graphs of magmas as introduced by Caucal [2].

Definition 3.11. Let V_R be a magma.

- The (*generalized*) *Cayley digraph* of V_R with respect to a subset $S \subseteq V$ is the digraph $\Gamma(V_R, S)$ with vertex set V and edge set

$$E := \{(v, R_s(v)) \mid v \in V, s \in S\} \subseteq V \times V.$$

  We say that S is the *connection set* of $\Gamma(V_R, S)$.

- The (*generalized*) *Cayley graph* of V_R with respect to S, denoted by $\Gamma_{\text{und}}(V_R, S)$, is the underlying (simple undirected) graph of $\Gamma(V_R, S)$.

- If $S = V$, then we call $\Gamma(V) := \Gamma(V_R, V)$ the *full Cayley digraph* of V_R. The *full Cayley graph* of V_R is defined similarly.

Remark 3.12. Unlike with Cayley graphs of *groups* as typically considered in the literature, we do not assume that the connection set S in Definition 3.11 is symmetric. That is, we do not assume that $\mathrm{id}_V \notin R(S)$ or that $R(S) = R(S)^{-1}$; this is consistent with the definitions of Cayley graphs of quandles given in, for example, [8, 16, 31].

We also do not assume that $R(S)$ is a generating subset of $\mathrm{RMlt}\, V_R$; this is consistent with the definitions given in, for example, [2, 3]. This is why we call the Cayley graphs in Definition 3.11 "generalized."

3.4 Schreier graphs

As Iwamoto et al. [16] note, Cayley (di)graphs of right quasigroups are special types of Schreier (di)graphs, which are important objects of study in combinatorial and geometric group theory.

Definition 3.13. Let T be a subset of a group G, and let V be a (left) G-set. The (*generalized*) *Schreier digraph* $\Gamma^{\mathrm{Sch}}(G, V, T)$ is the digraph with vertex set V and edge set

$$E := \{(v, t \cdot v) \mid v \in V,\ t \in T\} \subseteq V \times V.$$

The (*undirected*) *Schreier graph* $\Gamma^{\mathrm{Sch}}_{\mathrm{und}}(G, V, T)$ is defined similarly.

Remark 3.14. For all vertices $v, w \in V$ of $\Gamma := \Gamma^{\mathrm{Sch}}_{\mathrm{und}}(G, V, T)$, the pair $\{v, w\}$ is an edge of Γ if and only if there exists an element $t \in T$ such that $t \cdot w = v$ or $t^{-1} \cdot w = v$.

Remark 3.15. If T is a symmetric generating subset of G, then taking $V := G$ with the (left) regular action in Definition 3.13 recovers the traditional definition of the Cayley (di)graph of a group.

Remark 3.16. Given a right quasigroup V_R and a subset $S \subseteq V$, let $G := \mathrm{RMlt}\, V_R$ and $T := R(S)$. Then

$$\Gamma^{\mathrm{Sch}}(G, V, T) = \Gamma(V_R, S), \qquad \Gamma^{\mathrm{Sch}}_{\mathrm{und}}(G, V, T) = \Gamma_{\mathrm{und}}(V_R, S).$$

In the case that V_R is a quandle and $R(S)$ generates G, Iwamoto et al. called $\Gamma_{\mathrm{und}}(V_R, S)$ an *inner graph*; in recent work, they applied the above equality to study quandles using methods from geometric group theory [16, Section 3].

3.4.1 Preliminary results

Our solutions to Problem 1.2 can be stated nicely in terms of Schreier (di)graphs; cf. Theorems 6.1–6.2.

Proposition 3.17. *Let $\Gamma := \Gamma^{\mathrm{Sch}}(G, V, T)$ be a Schreier digraph with edge set E, and let H be a generating subset of G. The following are equivalent:*

1. *The action of G on V is also an action on Γ by digraph automorphisms.*

2. *For all elements $h \in H$, $v \in V$, and $s \in T$, there exists an element $t \in T$ such that*

$$th \cdot v = hs \cdot v. \tag{2}$$

Proof. (1) $\implies$ (2): Let $h \in H$, $s \in T$, and $v \in V$, so $(v, s \cdot v) \in E$. By assumption, $(h \cdot v, hs \cdot v) \in E$, so there exists an element $t \in T$ that satisfies Equation (2).

(2) $\implies$ (1): To show that G acts on Γ by digraph automorphisms, it suffices to show that $(h \cdot v, hs \cdot v) \in E$ for all elements $h \in H$ and directed edges $(v, s \cdot v) \in E$. By assumption, for all such elements and edges, there exists an element $t \in T$ that satisfies Equation (2). Hence, $(h \cdot v, hs \cdot v) \in E$. □

Proposition 3.18. *Let $\Gamma := \Gamma^{\mathrm{Sch}}_{\mathrm{und}}(G, V, T)$ be a Schreier graph, and let H be a generating subset of G. The following are equivalent:*

1. *The action of G on V is also an action on Γ by graph automorphisms.*

2. *For all elements $h \in H$, $v \in V$, and $s \in T$, there exists an element $t \in T$ such that one of the following equations holds:*

$$th \cdot v = hs \cdot v, \qquad h \cdot v = ths \cdot v.$$

Proof. The proof is nearly identical to that of Proposition 3.17; the only difference lies in using Remark 3.14 in the obvious ways. □

4 Motivating examples

We consider several examples of Cayley (di)graphs Γ of right quasigroups V_R and whether or not (Γ, R) is a marked graph; see also [1] and [8, Section 1.15]. These constructions serve as useful (counter)examples later in the paper.

Example 4.1. Equip the set $V = [3]$ with the right quasigroup structure given by

$$R_1 = \mathrm{id}_V, \quad R_2 = (23), \quad R_3 = (13).$$

By Lemma 2.14, V_R is not a rack because

$$R_3 R_2 R_3^{-1} = (12) \notin R(V).$$

Figure 1 depicts the full Cayley digraph $\Gamma(V_R)$ and the full Cayley graph $\Gamma_{\mathrm{und}}(V_R, V)$. Evidently, V_R is not realizable by either $\Gamma(V_R)$ or $\Gamma_{\mathrm{und}}(V_R, V)$; the only nontrivial (di)graph automorphism is (12). In particular, R is a marking of neither $\Gamma(V_R)$ nor $\Gamma_{\mathrm{und}}(V_R, V)$.

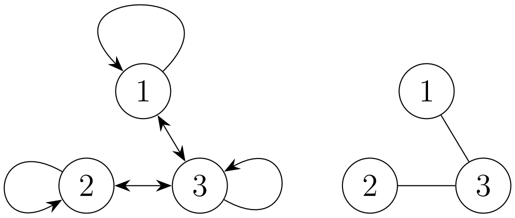

Figure 1: Full Cayley digraph and full Cayley graph of the right quasigroup from Example 4.1.

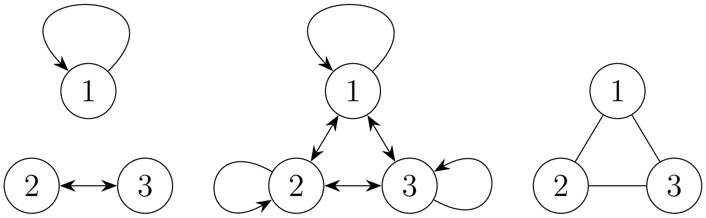

Figure 2: Partial and full Cayley digraphs and full Cayley graph of the quandle from Example 4.2.

Example 4.2. Equip the set $V = [3]$ with the quandle structure given by

$$R_1 = (23), \quad R_2 = (13), \quad R_3 = (12).$$

Note that V_R is a kei.

Let $S = \{1\}$. Figure 2 depicts the partial Cayley digraph $\Gamma(V_R, S)$, the full Cayley digraph $\Gamma(V_R)$, and the full Cayley graph $\Gamma_{\text{und}}(V_R, V)$. Although R is not a marking of $\Gamma(V_R, S)$ (or even $\Gamma_{\text{und}}(V_R, S)$), it is a marking of $\Gamma(V_R)$ and, hence, of $\Gamma_{\text{und}}(V_R, V)$.

Example 4.3. Nonisomorphic right quasigroups may share the same Cayley graphs and even the same Cayley digraphs. For example, let V be the set $[3]$. Figure 3 depicts the full Cayley digraph of the right quasigroup V_R defined by

$$R_1 = (12), \quad R_2 = (13), \quad R_3 = (23),$$

the full Cayley digraph of the permutation rack $V_{(123)}$, and the full Cayley graph shared by V_R and $V_{(123)}$. Evidently, V_R has the same full Cayley digraph as the quandle from Example 4.2. Moreover, V_R and $V_{(123)}$ have the same full Cayley graph. Of course, none of the right quasigroups in question are isomorphic; V_R is not a rack, $V_{(123)}$ is a non-involutory rack, and the quandle from Example 4.2 is a kei.

Example 4.4. Equip the set $V = [4]$ with the right quasigroup structure given by

$$R_1 = \mathrm{id}_V, \quad R_2 = (1234), \quad R_3 = (13)(24), \quad R_4 = (24).$$

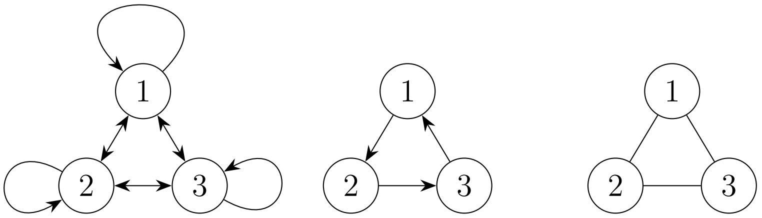

Figure 3: Full Cayley digraphs of the two right quasigroups from Example 4.3 and their shared full Cayley graph.

Note that

$$R_2R_4R_2^{-1} = (13) \notin R(V) \cup R(V)^{-1}.$$

In particular, Lemma 2.14 shows that V_R is not a rack.

Figure 4 depicts the full Cayley digraph $\Gamma := \Gamma(V_R)$ and the full Cayley graph $\Gamma_{\mathrm{und}}(V_R, V)$. Unlike in Example 4.1, R is a marking of Γ, so (Γ, R) realizes V_R^Γ. (In fact, not only does R land in $\operatorname{Aut}\Gamma$, but in fact $\operatorname{RMlt} V_R = \operatorname{Aut}\Gamma \cong D_4$.)

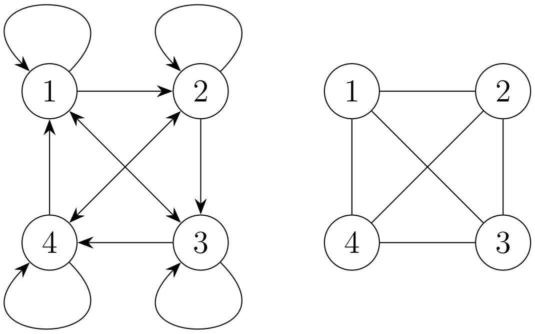

Figure 4: Full Cayley digraph and full Cayley graph of the right quasigroup from Example 4.4.

Example 4.5. Equip the set $V = [5]$ with the quandle structure given by

$$R_1 = (345), \quad R_2 = (354), \quad R_3 = (12)(45), \quad R_4 = (12)(35), \quad R_5 = (12)(34).$$

Let $S = \{1\}$. Figure 5 depicts the partial Cayley digraph $\Gamma = \Gamma(V_R, S)$ and its underlying graph $\Gamma_{\mathrm{und}}(V_R, S)$. Evidently, R is not a marking of $\Gamma(V_R, S)$, but it is a marking of $\Gamma_{\mathrm{und}}(V_R, S)$. (Indeed, the automorphism group of the former is $\langle (345), (12)\rangle \cong \mathbb{Z}/6\mathbb{Z}$, which does not contain $R(V)$, while the automorphism group of the latter is $\operatorname{RMlt} V_R \cong S_3 \times \mathbb{Z}/2\mathbb{Z}$.)

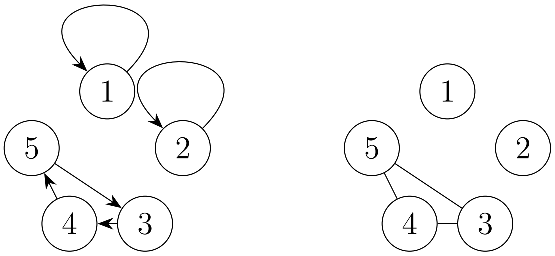

Figure 5: Partial Cayley digraph and underlying Cayley graph of the right quasigroup from Example 4.5.

5 From marked graphs to racks

In this section, we briefly deduce an answer to Problem 1.1. As an application, we introduce two rack-theoretic invariants of graphs.

5.1 Solution to Problem 1.1

Our alternative characterization of racks does the heavy lifting.

Theorem 5.1. *Let R be a marking (resp. q-marking) of a (di)graph Γ with vertex set V. Then V_R^Γ is a rack (resp. quandle) if and only if R is a magma homomorphism from V_R^Γ to* $\operatorname{Conj}(\operatorname{Aut}\Gamma)$*.*

Proof. The statement for markings follows immediately from Proposition 2.13. Thus, the statement about q-markings follows from Remark 3.5. ☐

Remark 5.2. Theorem 5.1 can be rephrased to state that (Γ, R) realizes a rack if and only if the group action of $\operatorname{Aut}\Gamma$ on $V(\Gamma)$ restricts to a *rack action* of $R(V(\Gamma))$ on $V(\Gamma)$; cf. [7].

Example 5.3. Let Γ be the complete digraph $\overrightarrow{K_3}$ of order 3 (with loops). Then Γ is the full Cayley digraph of the quandle from Example 4.2 and the non-rack right quasigroup from Example 4.3.

These two right quasigroups are constructed using the same permutations of $V = [3]$, all of which happen to be automorphisms of Γ. However, the different choices of vertices that $R : V_R^\Gamma \to \operatorname{Conj}(\operatorname{Aut}\Gamma)$ assigns to those automorphisms determine whether or not R is a magma homomorphism and, hence, whether or not V_R^Γ is a rack.

5.2 Application of Theorem 5.1

Given a (di)graph Γ, let $\mu_{\text{rack}}(\Gamma)$ (resp. $\mu_{\text{qnd}}(\Gamma)$) be the number of markings of Γ that realize racks (resp. quandles). As an application of Theorem 5.1, we compute these numbers for several graphs and discuss how to compute them in general. This is motivated by Problem 1.1.

In light of Remark 5.2, Theorem 5.1 immediately implies the following.

n	0	1	2	3	4	5	6	7
K_n	$(1,1)$	$(1,1)$	$(2,1)$	$(13,5)$	$(114,36)$	?	?	?
$K_{1,n-1}$	n/a	$(1,1)$	$(2,1)$	$(4,2)$	$(31,13)$	$(390,114)$	?	?
C_n	n/a	n/a	n/a	$(13,5)$	$(32,8)$	$(41,7)$	$(108,13)$	$(113,9)$

Table 1: Computations of $(\mu_{\text{rack}}, \mu_{\text{qnd}})$ for complete graphs K_n, star graphs $K_{1,n-1}$, and cycle graphs C_n for small values of n.

Corollary 5.4. *Let* Γ_1 *and* Γ_2 *be (di)graphs whose automorphism groups are isomorphic to a group* G*. If* $V(\Gamma_1) \cong V(\Gamma_2)$ *as* G*-sets, then* $(\mu_{\text{rack}}(\Gamma_1), \mu_{\text{qnd}}(\Gamma_1)) = (\mu_{\text{rack}}(\Gamma_2), \mu_{\text{qnd}}(\Gamma_2))$*.*

Corollary 5.5. *The numbers* μ_{rack} *and* μ_{qnd} *are (di)graph invariants.*

To compute $\mu_{\text{rack}}(\Gamma)$ given the action of $G := \operatorname{Aut}\Gamma$ on $V := V(\Gamma)$, Theorem 5.1 implies that it suffices to count the number of functions $R : V \to G$ that are magma homomorphisms from V_R to $\operatorname{Conj} G$. When $n := |V|$ is finite, this calculation is possible via a computer search. Namely, we use the `GRAPE` package [26] in `GAP` [13] to compute the image $\rho(G) \cong G$ of the permutation representation $\rho : G \hookrightarrow S_n$ under the identification $V = [n]$. To compute $\mu_{\text{rack}}(\Gamma)$ and $\mu_{\text{qnd}}(\Gamma)$, we go through all $|G|^n$ possible functions $R : V \to \rho(G)$ and count how many of the corresponding right quasigroup structures satisfy the rack and quandle axioms.

We provide an implementation of this exhaustive search algorithm in a GitHub repository [28]. With this implementation, we were able to compute μ_{rack} and μ_{qnd} for complete graphs K_n (equivalently, edgeless graphs), star graphs $K_{1,n-1}$, and cycle graphs C_n for small values of n; see Table 1. We also have the following general results for path graphs and cycle graphs.

Proposition 5.6. *Let* P_n *be a path graph of order* $n \geq 2$*. Then*

$$\mu_{\text{rack}}(P_n) = \begin{cases} 2^k & \text{if } n = 2k, \\ 2^{k+1} & \text{if } n = 2k+1, \end{cases} \qquad \mu_{\text{qnd}}(P_n) = \begin{cases} 1 & \text{if } n = 2k, \\ 2 & \text{if } n = 2k+1. \end{cases}$$

Proof. Let $V(P_n) = [n]$ be the vertices of P_n in order, that is, the edges of P_n are $(i, i+1)$ for all $1 \leq i \leq n-1$. Then the nonidentity element σ of $\operatorname{Aut} P_n \cong \mathbb{Z}/2\mathbb{Z}$ swaps i and $n+1-i$ for all $1 \leq i \leq \lfloor n/2 \rfloor$. Therefore, a function $R : V(P_n) \to \operatorname{Conj}(\operatorname{Aut} P_n)$ is a magma homomorphism if and only if $R_i = R_{n+1-i}$ for all $1 \leq i \leq n-1$. Since σ has no fixed points if $n = 2k$ and a unique fixed point when $n = 2k+1$, the claim follows from Theorem 5.1. □

Proposition 5.7. *Let* C_n *be a cycle graph of order* $n \geq 3$*, and let* $\sigma(n)$ *be the sum of all divisors of* n*. Then*

$$\mu_{\text{qnd}}(C_n) = \sigma(n) + 1.$$

The proof of Proposition 5.7 uses the geometric interpretation of reflections in the dihedral group $D_n \cong \operatorname{Aut} C_n$. Although the proof is not terribly long, we defer it to Appendix A to avoid interrupting the flow of the paper.

Example 5.8. When $n = 3$, five markings of the cycle graph C_3 realize quandles. In particular, let V_R be the quandle from Example 4.2. The full Cayley graph $\Gamma_{\text{und}}(V_R, V)$ (which is isomorphic to C_3) depicted in Figure 2, marked by the quandle structure $R : V \to S_3 = \operatorname{Aut} C_3$, realizes V_R. In the following section, we generalize this by showing that *all* racks are realized by their full Cayley (di)graphs.

6 From racks to marked graphs

In this section, we answer the second question in Problem 1.2. We start by addressing a generalized version of the question for right quasigroups and deduce solutions for racks afterward.

6.1 Results for right quasigroups

Propositions 3.17 and 3.18 do the heavy lifting in the directed and undirected cases, respectively.

Theorem 6.1. *Let S be a subset of a right quasigroup V_R, and let $\Gamma := \Gamma(V_R, S)$. The following are equivalent:*

1. *(Γ, R) is a marked digraph that realizes V_R.*
2. *R is a marking of Γ.*
3. *For all $h, v \in V$ and $s \in S$, there exists an element $t \in S$ such that*

$$R_t R_h(v) = R_h R_s(v).$$

Proof. (1) $\iff$ (2): Immediate.

(2) $\iff$ (3): By definition, $H := R(V)$ generates $G := \operatorname{RMlt} V_R$. In light of Remark 3.16, the equivalence of (2) and (3) is a special case of Proposition 3.17. ☐

Theorem 6.2. *Let S be a subset of a right quasigroup V_R, and let $\Gamma := \Gamma_{\text{und}}(V_R, S)$. The following are equivalent:*

1. *(Γ, R) is a marked graph that realizes V_R.*
2. *R is a marking of Γ.*
3. *For all $h, v \in V$ and $s \in S$, there exists an element $t \in S$ such that one of the following equations holds:*

$$R_t R_h(v) = R_h R_s(v), \qquad R_h(v) = R_t R_h R_s(v).$$

Proof. Similar to the proof of Theorem 6.1, with Proposition 3.18 in place of Proposition 3.17. ☐

Remark 6.3. The conditions of Theorem 6.2 are strictly weaker than those of Theorem 6.1. Indeed, Example 4.5 gives an example of a quandle V_R and a subset $S \subseteq V$ such that R is a marking of $\Gamma_{\mathrm{und}}(V_R, S)$ but not a marking of $\Gamma(V_R, S)$.

6.2 Specialization to racks

By considering the third conditions in Theorems 6.1–6.2, we answer the second question in Problem 1.2 in its original form: All racks are realizable by their full Cayley (di)graphs.

Corollary 6.4. *In the setting of Theorem 6.1 (resp. Theorem 6.2), if conjugating $R(S)$ by elements of $R(V)$ lands in $R(S)$ (resp. $R(S) \cup R(S)^{-1}$), then (Γ, R) is a marked digraph (resp. graph) realizing V_R. In particular, if V_R is a rack and Γ is its full Cayley (di)graph, then (Γ, R) realizes V_R.*

Proof. The conditions in the first claim directly imply the third conditions in Theorems 6.1–6.2. Therefore, the second claim follows from Lemma 2.14. ☐

Remark 6.5. If V_R is a right quasigroup but not a rack, then it is not true in general that conjugation in $R(V)$ lands in $R(V)$ (resp. $R(V) \cup R(V)^{-1}$). Nevertheless, the full Cayley digraph (resp. graph) may still satisfy the conditions of Theorem 6.1 (resp. Theorem 6.2); see Example 4.4.

Remark 6.6. Certainly, the full Cayley (di)graphs of non-rack right quasigroups do not satisfy the conditions of Theorems 6.1–6.2 in general; see Example 4.1.

Remark 6.7. The partial Cayley (di)graphs of quandles do not satisfy the conditions of Theorems 6.1–6.2 in general; see Examples 4.2 and 4.5.

7 Characterization of labeled Cayley digraphs

In this section, we give a graph-theoretic characterization of labeled Cayley digraphs of right-cancellative magmas, right-divisible magmas, right quasigroups, and certain classes of racks.

7.1 Preliminaries

First, we recall several definitions from nonassociative algebra and the theory of labeled digraphs.

7.1.1 Generalizations of right quasigroups

We recall two classes of magmas that generalize right quasigroups.

Definition 7.1. Let V_R be a magma. We say that V_R is *right-cancellative* (resp. *right-divisible*) if R_v is injective (resp. surjective) for all $v \in V$.

Remark 7.2. A magma is a right quasigroup if and only if it is both right-cancellative and right-divisible.

Remark 7.3. If V_R is a finite magma, then V_R is right-cancellative if and only if it is right-divisible.

Example 7.4. As in Example 2.4, the magma structure given by the addition maps $R_y(x) := x + y$ on the positive rational numbers $\mathbb{Q}^+$ yields a right-cancellative magma that is not right-divisible.

Example 7.5. Conversely, the magma structure given by $R_y(x) := x^3 - x$ on the set of rational numbers $\mathbb{Q}$ yields a right-divisible magma that is not right-cancellative.

7.1.2 Labeled digraphs

Following Caucal [2,3], we discuss labelings of digraph edges by vertices.

Definition 7.6.

- A *labeled digraph* is a triple $\Gamma = (V, E, L)$ where V is a set, $L \subseteq V$ is a subset, and $E \subseteq V \times L \times V$. Given $(v, \ell, w) \in E$, we say that ℓ is the *label* of the *edge* (v, ℓ, w). We say that V, L, and E are the *vertex, labeling,* and (*labeled*) *edge sets* of Γ, respectively.
- The *labeled Cayley digraph* of a magma V_R with respect to a *connection set* $S \subseteq V$, denoted by $\Gamma_{\text{lab}}(V_R, S)$, is the labeled digraph (V, E, S) in which
$$E = \{(v, s, R_s(v)) \mid v \in V,\ s \in S\}.$$

Remark 7.7. Labeled edges $(v, \ell, w) \in E$ can also be denoted by *transitions* $v \xrightarrow{\ell} w$; for example, see Figures 6–8.

Definition 7.8. Let $\Gamma = (V, E, L)$ be a labeled digraph.

- Let $\pi_1 : E \to V \times L$ and $\pi_2 : E \to L \times V$ be the projections from E onto its first two and last two coordinates, respectively.
- We say that Γ is *deterministic* (resp. *codeterministic*) if π_1 (resp. π_2) is injective.
- We say that Γ is *source-complete* or *executable* (resp. *target-complete* or *coexecutable*) if π_1 (resp. π_2) is surjective.

Let $\mathcal{D}$ denote the class of deterministic, source-complete labeled digraphs.

The following is immediate.

Remark 7.9. Let V_R be a magma, let $S \subseteq V$, and let $\Gamma = \Gamma_{\text{lab}}(V_R, S)$. Then:

- Γ lies in $\mathcal{D}$.
- If V_R is right-cancellative, then Γ is codeterministic.
- If V_R is right-divisible, then Γ is target-complete.

Our first objective will be to prove a converse to Remark 7.9; see Proposition 7.11.

7.2 Construction of V_R^Γ

Given an element $\Gamma = (V, E, L)$ of $\mathcal{D}$, we define a magma structure R on V as follows. Since we do not consider marked graphs for the remainder of the paper, we denote this magma by V_R^Γ, overwriting the notation from previous sections.

For each non-label vertex $v \in V \setminus L$, let $R_v := \mathrm{id}_V$. Otherwise, for each label $\ell \in L$, define $R_\ell : V \to V$ as follows. Given a vertex $v \in V$, the preimage $\pi_1^{-1}(v, \ell)$ contains a unique element (v, ℓ, w) because Γ is deterministic and source-complete. Thus, define $R_\ell(v) := w$. The assignment $v \mapsto R_v$ makes V into a magma V_R^Γ. Verifying the following is straightforward.

Remark 7.10. If $\Gamma = (V, E, L)$ is an element of $\mathcal{D}$, then Γ is the labeled Cayley digraph $\Gamma_{\text{lab}}(V_R^\Gamma, L)$.

7.3 First results

We apply our construction of V_R^Γ.

Proposition 7.11. *Let $\Gamma = (V, E, L)$ be an element of $\mathcal{D}$.*

1. *If Γ is codeterministic, then V_R^Γ is right-cancellative.*
2. *If Γ is target-complete, then V_R^Γ is right-divisible.*

Proof. (1): Suppose that Γ is codeterministic, so π_2 is injective. We have to show for all $\ell \in V$ that R_ℓ is injective. If $\ell \notin L$, we are done. Otherwise, suppose for some $v, w \in V$ that

$$R_\ell(v) = R_\ell(w) =: x.$$

Since π_2 is injective, $\pi_2^{-1}(\ell, x)$ contains at most one element. Since $\pi_2^{-1}(\ell, x)$ contains (v, ℓ, x) and (w, ℓ, x), we obtain $v = w$, as desired.

(2): Suppose that Γ is target-complete, so π_2 is surjective. We have to show for all $\ell \in V$ that R_ℓ is surjective. If $\ell \notin L$, we are done. Otherwise, let $w \in V$. By hypothesis, $\pi_2^{-1}(\ell, w)$ contains an edge of Γ, say (v, ℓ, w). Hence, $R_\ell(v) = w$. ☐

Figure 6: The *first rack condition* from Definition 7.13 states that for all subgraphs of the form on the left, there also exists a subgraph of the form on the right, where $x := R_{\ell_1}(w_2)$.

Henceforth, let $\mathcal{Q}$ denote the subclass of $\mathcal{D}$ whose elements are also codeterministic and target-complete. The following answers Problem 1.7 for right-cancellative magmas, right-divisible magmas, and right quasigroups.

Theorem 7.12. *A labeled digraph is the labeled Cayley digraph of a right-cancellative magma (resp. right-divisible magma, right quasigroup) if and only if it is an element of $\mathcal{D}$ that is codeterministic (resp. target-complete, contained in $\mathcal{Q}$).*

Proof. The first two claims follow from Remarks 7.9–7.10 and Proposition 7.11. Therefore, the third claim follows from Remark 7.2. ☐

7.4 Specialization to racks

Next, we specialize Theorem 7.12 to racks, quandles, involutory right quasigroups, involutory racks, and kei. To that end, we introduce several graph-theoretic conditions corresponding to the labeled Cayley digraphs of objects in these categories.

Recall from Remark 7.10 and Proposition 7.11 that each element $\Gamma = (V, E, L)$ in $\mathcal{Q}$ is the labeled Cayley digraph of the right quasigroup V_R^Γ with respect to the connection set L.

Definition 7.13. Let $\Gamma = (V, E, L)$ be an element of $\mathcal{Q}$.

- We say that Γ satisfies the *first rack condition* if

$$R_{\ell_1}(w_2) = R_{R_{\ell_1}(\ell_2)}(w_1)$$

  for all $v \in V$ such that $(v, \ell_1, w_1), (v, \ell_2, w_2) \in E$. See Figure 6 for a visualization. Note that we do not assume that $w_1 \neq w_2$.

- We say that Γ satisfies the *second rack condition* if for all edges (v, ℓ, w) and non-label vertices $x \in V \setminus L$ such that $R_\ell(x) \in L$ is a label, then w has a loop labeled by $R_\ell(x)$; i.e.,

$$(w, R_\ell(x), w) \in E.$$

  See Figure 7 for a visualization.

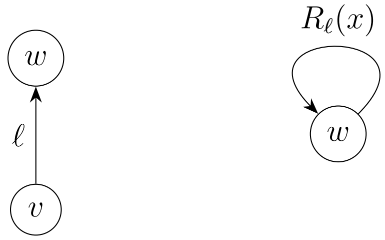

Figure 7: The *second rack condition* from Definition 7.13 states that for all subgraphs of the form on the left and non-label vertices $x \in V \setminus L$, if $R_\ell(x) \in L$ is a label, then there exists a subgraph of the form on the right.

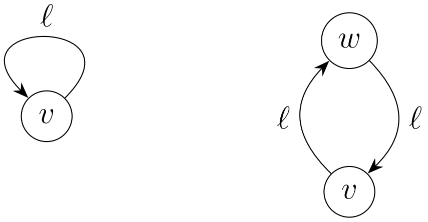

Figure 8: The *label-involutory* condition from Definition 7.13 states that subgraphs of the forms on the left and the right partition the edge set.

- We say that Γ is *label-idempotent* if $R_\ell(\ell) = \ell$ for all $\ell \in L$; that is, each vertex ℓ contained in the labeling set L has a loop $(\ell, \ell, \ell) \in E$ labeled by ℓ.
- We say that Γ satisfies the *label-involutory* if $R_\ell^2(v) = v$ for all $v \in V$ and $\ell \in L$; that is, the edge set E can be partitioned into loops and cycles of length 2 having the form
$$\{(v, \ell, w), (w, \ell, v)\}.$$
  See Figure 8 for a visualization.

7.4.1 Result

We answer Problem 1.7 for racks, quandles, and involutory right quasigroups. This also yields answers for involutory racks and kei in the obvious way.

Theorem 7.14. *Let $\Gamma = (V, E, L)$ be a labeled digraph.*

1. *Γ is the labeled Cayley digraph of a rack if and only if Γ lies in $\mathcal{Q}$ and satisfies the two rack conditions.*
2. *Γ is the labeled Cayley digraph of a quandle if and only if Γ lies in $\mathcal{Q}$, satisfies the two rack conditions, and is label-idempotent.*
3. *Γ is the labeled Cayley digraph of an involutory right quasigroup if and only if Γ is a label-involutory element of $\mathcal{Q}$.*

Proof. By Theorem 7.12, we can assume that $\Gamma \in \mathcal{Q}$. As noted before, this inclusion implies that $\Gamma = \Gamma_{\mathrm{lab}}(V_R^\Gamma, L)$.

(1): First, suppose that V_R^Γ is a rack; we show that Γ satisfies the rack conditions. For all vertices $v \in V$ and edges $(v, \ell_i, w_i) \in E$, we have $w_i = R_{\ell_i}(v)$. It follows from Equation (1) that

$$R_{\ell_1}(w_2) = R_{\ell_1}R_{\ell_2}(v) = R_{R_{\ell_1}(\ell_2)}R_{\ell_1}(v) = R_{R_{\ell_1}(\ell_2)}(w_1).$$

Hence, Γ satisfies the first rack condition. Next, let $(v, \ell, w) \in E$, and let $x \in V \setminus L$ satisfy $R_\ell(x) \in L$. Since $R_x = \mathrm{id}_V$ and $R_\ell(v) = w$, Equation (1) yields

$$R_{R_\ell(x)}(w) = R_{R_\ell(x)}R_\ell(v) = R_\ell R_x(v) = R_\ell(v) = w.$$

Since $\Gamma = \Gamma_{\mathrm{lab}}(V_R^\Gamma, L)$, it follows that $(w, R_\ell(x), w) \in E$, so Γ satisfies the second rack condition.

Conversely, suppose that $\Gamma \in \mathcal{Q}$ satisfies the two rack conditions; we show that V_R^Γ is a rack. We have to verify that

$$R_{\ell_1}R_{\ell_2}(v) = R_{R_{\ell_1}(\ell_2)}R_{\ell_1}(v)$$

for all $\ell_1, \ell_2, v \in V$. If $\ell_1 \notin L$, we are done. Next, suppose that $\ell_1 \in L$ and $\ell_2 \notin L$. If $R_{\ell_1}(\ell_2) \notin L$, we are done. Otherwise, applying the second rack condition to the edge $(v, \ell_1, R_{\ell_1}(v))$ yields the desired equality. Finally, if $\ell_1, \ell_2 \in L$, then $(v, \ell_i, w_i) \in E$ with $w_i := R_{\ell_i}(v)$. Since Γ satisfies the first rack property,

$$R_{\ell_1}R_{\ell_2}(v) = R_{\ell_1}(w_2) = R_{R_{\ell_1}(\ell_2)}(w_1) = R_{R_{\ell_1}(\ell_2)}R_{\ell_1}(v).$$

(2): By the previous claim, it suffices to show that V_R^Γ is a quandle if and only if Γ satifies the label-idempotence condition. But this is clear from the construction of V_R^Γ.

(3): Clear from the construction of V_R^Γ. □

Corollary 7.15. *Let $\mathcal{I}$ be the subclass of $\mathcal{Q}$ whose elements are label-involutory and satisfy the two rack conditions. Then $\mathcal{I}$ (resp. the subclass of $\mathcal{I}$ whose elements are label-idempotent) is precisely the class of labeled Cayley digraphs of involutory racks (resp. kei).*

8 Open questions

We conclude by proposing directions for future work. First, Problem 1.1 motivates the following.

Problem 8.1. Compute μ_{rack} and μ_{qnd} for more families of (di)graphs.

Problem 8.2. Add more entries to Table 1.

A more computationally efficient implementation of the algorithm described in Subsection 5.2 will help in addressing Problems 8.1–8.2.

The existence of nonisomorphic graphs that satisfy the hypotheses of Corollary 5.4—for example, any non-self-complementary graph Γ and its complement $\overline{\Gamma}$—shows that the pair $(\mu_{\text{rack}}, \mu_{\text{qnd}})$ is not a complete invariant of graphs. In this light, it is interesting to ask the following.

Problem 8.3. Under what conditions do nonisomorphic graphs share the same values of μ_{rack} and/or μ_{qnd} without satisfying the hypotheses of Corollary 5.4?

Finally, Problem 1.7 and recent work applying geometric group theory to quandle theory [16, 18, 20, 24] motivate further analogues of Theorem 7.14.

Problem 8.4. Characterize labeled Cayley digraphs of various classes of racks (e.g., medial racks, Latin quandles, fundamental racks of framed links).

Problem 8.5. Characterize unlabeled Cayley graphs of right quasigroups, racks, and quandles.

Problem 8.6. Define and characterize Cayley (di)graphs of classes of racks equipped with extra structure (e.g., generalized Legendrian racks [27], multi-virtual quandles [19], symmetric racks [29]).

A Proof of Proposition 5.7

A.1 Preliminaries

Let $n \geq 3$. Recall that a subgroup of the dihedral group D_n is called a *reflection subgroup* if it is either the trivial subgroup or generated by reflections.

Recall that the automorphism group of the cycle graph C_n is isomorphic to D_n. Therefore, by a 1975 result of Cavior [4], proving the following will also prove Proposition 5.7.

Proposition A.1. *Let $\mathcal{S}$ be the set of reflection subgroups of D_n, and let $\mathcal{M}$ be the set of markings R of C_n such that $V_R^{C_n}$ is a quandle. Then there exists a bijection $\varphi : \mathcal{S} \to \mathcal{M}$.*

A.2 Construction of φ

We construct a function $\varphi : \mathcal{S} \to \mathcal{M}$ geometrically. Given a reflection subgroup $G \in \mathcal{S}$, let $\varphi(G)$ be the marking $R : V \to D_n$ defined as follows. For each vertex $v \in V$, if v lies on the axis of a reflection $\psi \in G$ (which is necessarily unique if it exists), then let $R_v := \psi$. Otherwise, let $R_v := \mathrm{id}_V$.

Lemma A.2. *If $G \in \mathcal{S}$, then $\varphi(G) \in \mathcal{M}$.*

Proof. By construction, $R_v(v) = v$ for all $v \in V$. It remains to show that R is a rack structure on V. That is, we have to show that

$$R_v R_w R_v^{-1} = R_{R_v(w)}$$

for all vertices $v, w \in V$. But this is geometrically clear: Since R_w is either the identity map or the reflection about the axis ℓ containing w, the composition $R_v R_w R_v^{-1}$ is either the identity map or the reflection about the axis $R_v(\ell)$, i.e., the axis containing $R_v(w)$. This transformation is precisely $R_{R_v(w)}$. □

A.3 Bijectivity of φ

We construct an inverse map $\varphi^{-1} : \mathcal{M} \to \mathcal{S}$ as follows. Given a quandle structure $R \in \mathcal{M}$, each right-multiplication map R_v is either the identity map or a reflection. This is because every rotation in D_n has no fixed points. Therefore, defining

$$\varphi^{-1}(R) := \operatorname{RMlt} V_R = \langle R_v \mid v \in V \rangle$$

yields a function $\varphi^{-1} : \mathcal{M} \to \mathcal{S}$. Verifying that φ and φ^{-1} are mutually inverse is straightforward. This completes the proof of Proposition A.1 and, hence, that of Proposition 5.7.

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Tensor products of Leibniz bimodules and Grothendieck rings

Jörg Feldvoss and Friedrich Wagemann

Abstract. In this paper we define three different notions of tensor products for Leibniz bimodules. The "natural" tensor product of Leibniz bimodules is not always a Leibniz bimodule. In order to fix this, we introduce the notion of a weak Leibniz bimodule and show that the "natural" tensor product of weak bimodules is again a weak bimodule. Moreover, it turns out that weak Leibniz bimodules are modules over a cocommutative Hopf algebra canonically associated with the Leibniz algebra. Therefore, the category of all weak Leibniz bimodules is symmetric monoidal and the full subcategory of finite-dimensional weak Leibniz bimodules is rigid and pivotal. On the other hand, we introduce two truncated tensor products of Leibniz bimodules, which are again Leibniz bimodules. These tensor products induce a non-associative multiplication on the Grothendieck group of the category of finite-dimensional Leibniz bimodules. In particular, we prove that in characteristic zero for a finite-dimensional solvable Leibniz algebra over an algebraically closed field, this Grothendieck ring is an alternative power-associative commutative Jordan ring, but for a finite-dimensional non-zero semi-simple Leibniz algebra, it is neither alternative nor a Jordan ring.

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Contents

1 Introduction

Leibniz algebras (see [15–17]) are non-associative algebras which generalize Lie algebras in the sense that a Leibniz algebra has a bilinear multiplication which is not necessarily anti-commutative, but whose left or right multiplication operators are derivations. A *Leibniz* $\mathfrak{L}$*-bimodule* M is a Beck module over a Leibniz algebra $\mathfrak{L}$ described abstractly as an abelian group object in the slice category or, more concretely, as the structure we obtain on the vector space M when we declare $\mathfrak{L} \oplus M$ to be a Leibniz algebra, or more explicitly, a two-sided module M whose left and right $\mathfrak{L}$-actions are bilinear and satisfy the conditions

$$(xy) \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m) \tag{LLM}$$
$$(x \cdot m) \cdot y = x \cdot (m \cdot y) - m \cdot (xy) \tag{LML}$$
$$(m \cdot x) \cdot y = m \cdot (xy) - x \cdot (m \cdot y) \tag{MLL}$$

for all elements $x, y \in \mathfrak{L}$ and $m \in M$ (see [17] for right Leibniz algebras and [8] for left Leibniz algebras which we will consider in this paper).

The category of modules over a Lie algebra admits a natural tensor product over the ground field. In this paper we ask how to define a similar tensor product for bimodules over a Leibniz algebra $\mathfrak{L}$. This is not obvious as $\mathfrak{L}$-bimodules are (right or left) modules over the universal enveloping algebra of $\mathfrak{L}$ (see [17]), but it is not known whether the latter is a bialgebra, and our paper seems to indicate that this might not be the case. It turns out that the tensor product $M \otimes N$ of $\mathfrak{L}$-bimodules M and N endowed with the left $\mathfrak{L}$-action

$$x \cdot (m \otimes n) = (x \cdot m) \otimes n + m \otimes (x \cdot n)$$

and the right $\mathfrak{L}$-action

$$(m \otimes n) \cdot x = (m \cdot x) \otimes n + m \otimes (n \cdot x)$$

satisfies the conditions (LLM) and (LML), but in general not the condition (MLL) (Proposition 3.1 and Example 3.10).

Based on this result, we pursue two lines of thoughts to define a tensor product on some category of bimodules over a Leibniz algebra. First of all, in Section 3, we introduce the concept of a *weak Leibniz bimodule* as a two-sided module over a Leibniz algebra satisfying (LLM) and (LML), but not necessarily (MLL). Then we show that weak Leibniz bimodules are the left modules over a cocommutative Hopf algebra canonically associated with the Leibniz algebra, its *weak universal enveloping algebra* (Theorem 3.12). We deduce from here that the category $\mathbf{Mod}_{\text{weak}}^{\text{bi}}(\mathfrak{L})$ of weak $\mathfrak{L}$-bimodules is a symmetric monoidal category (Theorem 3.15) and that its full subcategory $\mathbf{mod}_{\text{weak}}^{\text{bi}}(\mathfrak{L})$ of finite-dimensional weak $\mathfrak{L}$-bimodules is rigid and pivotal (Theorem 3.24 and Remark 3.25). In fact, $\mathbf{mod}_{\text{weak}}^{\text{bi}}(\mathfrak{L})$ is even a ring category (Theorem 3.26) in the sense of [5]. Along the way, we address the question of classifying the irreducible objects in $\mathbf{Mod}_{\text{weak}}^{\text{bi}}(\mathfrak{L})$ and obtain a partial answer (Proposition 3.9), we discuss the relation between the category $\mathbf{Mod}_{\text{weak}}^{\text{bi}}(\mathfrak{L})$ and the full subcategory $\mathbf{Mod}^{\text{bi}}(\mathfrak{L})$ of all $\mathfrak{L}$-bimodules (Proposition 3.17), and we give some examples designed to illustrate our results.

The second line of thought is to consider the tensor product $M \otimes N$ of the Leibniz bimodules M and N modulo a certain subspace in order to obtain an ordinary Leibniz bimodule. We look at two such subspaces which thus lead to two *truncated* tensor products $M \overline{\otimes} N$ and $M \underline{\otimes} N$ whose properties are discussed in Section 4. These truncated tensor products are the same when one tensor factor is symmetric or anti-symmetric (Theorem 4.6), and they coincide with the "natural" tensor product defined in Section 3 in the case that both factors are symmetric or anti-symmetric (Corollary 4.7). On the other hand, the truncated tensor products are zero when both factors are non-trivial irreducible, one factor is symmetric and the other one is anti-symmetric (Corollary 4.9). We give examples to show that in many cases both truncated tensor products are non-zero. But an important property which one looses by considering the truncated tensor products is the associativity.

In order to capture this loss of associativity more precisely, we construct in Section 5 the Grothendieck ring associated with the truncated tensor products and describe its algebraic structure. Note that we do not know whether $\overline{\otimes}$ and $\underline{\otimes}$ do coincide, but they always induce the same multiplication on the Grothendieck group $\mathrm{Gr}^{\text{bi}}(\mathfrak{L})$ of the category $\mathbf{mod}^{\text{bi}}(\mathfrak{L})$ of finite-dimensional $\mathfrak{L}$-bimodules for a Leibniz algebra $\mathfrak{L}$. This Grothendieck ring is a unital commutative ring (Proposition 5.1), but it is not necessarily associative. One of the main results in this section, Theorem 5.4, is the description of the Grothendieck ring $\mathrm{Gr}^{\text{bi}}(\mathfrak{L})$ as a so-called *unital commutative product* of two copies of the Grothendieck ring $\mathrm{Gr}(\mathfrak{L}_{\text{Lie}})$ of the canonical Lie algebra $\mathfrak{L}_{\text{Lie}}$ associated with the Leibniz algebra $\mathfrak{L}$. Here one copy of $\mathrm{Gr}(\mathfrak{L}_{\text{Lie}})$ corresponds to the classes of the symmetric irreducible $\mathfrak{L}$-bimodules and the other copy corresponds to the classes of the anti-symmetric irreducible $\mathfrak{L}$-bimodules each of which can be constructed in a natural way from the irreducible $\mathfrak{L}_{\text{Lie}}$-modules. We use this result to completely determine the structure of $\mathrm{Gr}^{\text{bi}}(\mathfrak{L})$ for a finite-dimensional solvable Leibniz algebra $\mathfrak{L}$ over an algebraically closed field of characteristic zero (Corollary 5.8), and we also prove that in this case $\mathrm{Gr}^{\text{bi}}(\mathfrak{L})$ is an alternative power-associative Jordan ring (Theorem 5.13 and Corollary 5.17). In addition, we give an example to show that Theorem 5.13 is not true for ground fields of non-zero characteristic (Example 5.14). On the other hand, we prove in Proposition 5.10 that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is not associative for those Leibniz algebras $\mathfrak{L}$ whose canonical Lie algebras $\mathfrak{L}_{\mathrm{Lie}}$ are finite dimensional. More specifically, we show that the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ of a finite-dimensional non-zero semi-simple Leibniz algebra $\mathfrak{L}$ over a field of characteristic zero is neither alternative nor a Jordan ring (Theorem 5.19).

Although we are able to obtain several results for (truncated) tensor products of Leibniz bimodules, many questions still remain open. Contrary to irreducible Leibniz bimodules, a classification of the irreducible weak Leibniz bimodules (up to isomorphism) is not known. Such a classification might also shed some light on the precise relationship between the categories $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ and $\mathbf{Mod}^{\mathrm{bi}}(\mathfrak{L})$ of (weak) $\mathfrak{L}$-bimodules or the Grothendieck rings of the corresponding subcategories $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ and $\mathbf{mod}^{\mathrm{bi}}(\mathfrak{L})$ of finite-dimensional (weak) $\mathfrak{L}$-bimodules. Moreover, except for finite-dimensional solvable Leibniz algebras over an algebraically closed field of characteristic zero and $\mathfrak{sl}_2(\mathbb{C})$, we do not know whether the Grothendieck ring of the category of finite-dimensional Leibniz bimodules is power-associative. Finally, it would be very interesting to explicitly compute the Grothendieck rings for non-solvable Leibniz algebras whose canonical Lie algebra is not isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ or for Leibniz algebras over a field of non-zero characteristic.

We finish this introduction by fixing some notation and conventions. A ring without any specification is an abelian group with a biadditive multiplication that not necessarily satisfies any other identities. Similarly, an algebra without any specification is a vector space with a bilinear multiplication that not necessarily satisfies any other identities. The multiplicative identity element of a ring or an algebra is said to be its unity, a ring or an algebra with a unity is called unital, and homomorphisms between unital rings or unital algebras preserve the unity. Ideals are always two-sided ideals if not explicitly stated otherwise. Modules over Lie algebras are always left modules. All vector spaces and algebras are defined over an arbitrary field which is only explicitly mentioned when some additional assumptions on the ground field are made or this enhances the understanding of the reader. In particular, with one exception, when we consider tensor products over the integers, which will be denoted by $\otimes_{\mathbb{Z}}$, all tensor products are over the relevant ground field and will be denoted by $\otimes$. For a subset X of a vector space V over a field $\mathbb{F}$ we let $\langle X\rangle_{\mathbb{F}}$ be the subspace of V spanned by X. We denote the space of linear transformations from an $\mathbb{F}$-vector space V to an $\mathbb{F}$-vector space W by $\mathrm{Hom}_{\mathbb{F}}(V,W)$. In particular, $\mathrm{End}_{\mathbb{F}}(V) := \mathrm{Hom}_{\mathbb{F}}(V,V)$ is the space of linear operators on V, and the linear dual $V^* := \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ is the space of linear forms on V. Finally, let id_X denote the identity function on a set X, let $\mathbb{N}_0$ be the set of non-negative integers, and let $\mathbb{Z}$ be the ring of integers.

2 Preliminaries

In this section we recall some definitions and prove a result about the universal enveloping algebra of a left Leibniz algebra that will be useful later in the paper.

A *left Leibniz algebra* is an algebra $\mathfrak{L}$ such that every left multiplication operator $L_x : \mathfrak{L} \to \mathfrak{L}$, $y \mapsto xy$ is a derivation. This is equivalent to the identity

$$x(yz) = (xy)z + y(xz)$$

for all elements $x, y, z \in \mathfrak{L}$, which in turn is equivalent to the identity

$$(xy)z = x(yz) - y(xz)$$

for all elements $x, y, z \in \mathfrak{L}$. There is a similar definition of a *right Leibniz algebra*, which is used by Loday et al. (see [15–17]), but as in our previous papers (see [8, 9, 11]), we will consider left Leibniz algebras which most of the time are just called Leibniz algebras.

Note that every Lie algebra is a left and a right Leibniz algebra. On the other hand, every Leibniz algebra has an important ideal, its Leibniz kernel, that measures how much the Leibniz algebra deviates from being a Lie algebra. Namely, let $\mathfrak{L}$ be a Leibniz algebra over a field $\mathbb{F}$. Then

$$\operatorname{Leib}(\mathfrak{L}) := \langle x^2 \mid x \in \mathfrak{L} \rangle_{\mathbb{F}}$$

is called the *Leibniz kernel* of $\mathfrak{L}$. The Leibniz kernel $\operatorname{Leib}(\mathfrak{L})$ is an abelian ideal of $\mathfrak{L}$, and $\operatorname{Leib}(\mathfrak{L}) \neq \mathfrak{L}$ whenever $\mathfrak{L} \neq 0$ (see [8, Proposition 2.20]). Moreover, $\mathfrak{L}$ is a Lie algebra if, and only if, $\operatorname{Leib}(\mathfrak{L}) = 0$.

By definition of the Leibniz kernel, $\mathfrak{L}_{\mathrm{Lie}} := \mathfrak{L}/\operatorname{Leib}(\mathfrak{L})$ is a Lie algebra which we call the *canonical Lie algebra* associated with $\mathfrak{L}$. In fact, the Leibniz kernel is the smallest ideal such that the corresponding factor algebra is a Lie algebra (see [8, Proposition 2.22]).

Next, we will briefly discuss left modules and bimodules of left Leibniz algebras. Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$. A *left $\mathfrak{L}$-module* is a vector space M over $\mathbb{F}$ with an $\mathbb{F}$-bilinear left $\mathfrak{L}$-action $\mathfrak{L} \times M \to M$, $(x, m) \mapsto x \cdot m$ such that

$$(xy) \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m)$$

is satisfied for every $m \in M$ and all $x, y \in \mathfrak{L}$. Note that for a Lie algebra a left Leibniz module is the same as a Lie module.

Moreover, every left $\mathfrak{L}$-module M gives rise to a homomorphism $\lambda : \mathfrak{L} \to \mathfrak{gl}(M)$ of left Leibniz algebras, defined by $\lambda_x(m) := x \cdot m$, and vice versa. We call λ the *left representation* of $\mathfrak{L}$ associated with M.

The correct concept of a module for a left Leibniz algebra $\mathfrak{L}$ is the notion of a Leibniz $\mathfrak{L}$-bimodule. An *$\mathfrak{L}$-bimodule* is a vector space M that is endowed with an $\mathbb{F}$-bilinear left $\mathfrak{L}$-action and an $\mathbb{F}$-bilinear right $\mathfrak{L}$-action satisfying the following compatibility conditions:

$$(xy) \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m) \tag{LLM}$$

$$(x \cdot m) \cdot y = x \cdot (m \cdot y) - m \cdot (xy) \tag{LML}$$

$$(m \cdot x) \cdot y = m \cdot (xy) - x \cdot (m \cdot y) \tag{MLL}$$

for all elements $x, y \in \mathfrak{L}$ and $m \in M$.

It is an immediate consequence of (LLM) that every Leibniz bimodule is a left Leibniz module.

Remark 2.1. Note that when (LML) holds, the relation (MLL) is equivalent to

$$(x \cdot m + m \cdot x) \cdot y = 0 \tag{ZD}$$

for all elements $x, y \in \mathfrak{L}$ and $m \in M$.

On the other hand, a pair (λ, ρ) of linear transformations $\lambda : \mathfrak{L} \to \operatorname{End}_{\mathbb{F}}(V)$ and $\rho : \mathfrak{L} \to \operatorname{End}_{\mathbb{F}}(V)$ is called a *representation* of $\mathfrak{L}$ on the $\mathbb{F}$-vector space V if the following conditions are satisfied:

$$\lambda_{xy} = \lambda_x \circ \lambda_y - \lambda_y \circ \lambda_x \tag{1}$$

$$\rho_{xy} = \lambda_x \circ \rho_y - \rho_y \circ \lambda_x \tag{2}$$

$$\rho_y \circ (\lambda_x + \rho_x) = 0 \tag{3}$$

for all elements $x, y \in \mathfrak{L}$. Then every $\mathfrak{L}$-bimodule M gives rise to a representation (λ, ρ) of $\mathfrak{L}$ on M via $\lambda_x(m) := x \cdot m$ and $\rho_x(m) := m \cdot x$. Conversely, every representation (λ, ρ) of $\mathfrak{L}$ on the vector space M defines an $\mathfrak{L}$-bimodule structure on M via $x \cdot m := \lambda_x(m)$ and $m \cdot x := \rho_x(m)$.

By virtue of [8, Lemma 3.3], every left $\mathfrak{L}$-module is an $\mathfrak{L}_{\mathrm{Lie}}$-module in a natural way, and vice versa. Consequently, many properties of left Leibniz modules follow from the corresponding properties of modules for the canonical Lie algebra.

The usual definitions of the notions of *sub(bi)module*, *irreducibility*, *complete reducibility*, *composition series*, *homomorphism*, *isomorphism*, etc., hold for left Leibniz modules and Leibniz bimodules. (Note that by definition an irreducible (bi)module is always non-zero.)

Let M be an $\mathfrak{L}$-bimodule for a left Leibniz algebra $\mathfrak{L}$. Then M is said to be *symmetric* if $m \cdot x = -x \cdot m$ for every $x \in \mathfrak{L}$ and every $m \in M$, and M is said to be *anti-symmetric* if $m \cdot x = 0$ for every $x \in \mathfrak{L}$ and every $m \in M$. Moreover, an $\mathfrak{L}$-bimodule M is called *trivial* if $x \cdot m = 0 = m \cdot x$ for every $x \in \mathfrak{L}$ and every $m \in M$. Note that an $\mathfrak{L}$-bimodule M is trivial if, and only if, M is symmetric and anti-symmetric. We call

$$M_0 := \langle x \cdot m + m \cdot x \mid x \in \mathfrak{L}, m \in M \rangle_{\mathbb{F}}$$

the *anti-symmetric kernel* of M. It is well known that M_0 is an anti-symmetric $\mathfrak{L}$-subbimodule of M such that $M_{\mathrm{sym}} := M/M_0$ is symmetric (see [8, Propositions 3.12 and 3.13]). Recall that every left $\mathfrak{L}$-module M of a left Leibniz algebra $\mathfrak{L}$ determines a unique symmetric $\mathfrak{L}$-bimodule structure on M by defining $m \cdot x := -x \cdot m$ for every element $m \in M$ and every element $x \in \mathfrak{L}$ (see [8, Proposition 3.15 (b)]). We call this $\mathfrak{L}$-bimodule the *symmetrization* of M which will be denoted by M^s. Similarly, every left $\mathfrak{L}$-module M with trivial right action is an anti-symmetric $\mathfrak{L}$-bimodule (see [8, Proposition 3.15 (a)]). We call this $\mathfrak{L}$-bimodule the *anti-symmetrization* of M which will be denoted by M^a.

Loday and Pirashvili [17, Section 2] define a universal enveloping algebra of a right Leibniz algebra. We will briefly recall the analogous construction for left Leibniz algebras. For any left Leibniz algebra $\mathfrak{L}$ over a field $\mathbb{F}$ we denote by $\mathfrak{L}^\ell$ an isomorphic copy of the underlying $\mathbb{F}$-vector space of $\mathfrak{L}$ (by the isomorphism $\ell : \mathfrak{L} \to \mathfrak{L}^\ell$), and we denote by $\mathfrak{L}^r$ another isomorphic copy of the underlying $\mathbb{F}$-vector space of $\mathfrak{L}$ (by the isomorphism $r : \mathfrak{L} \to \mathfrak{L}^r$). Then the tensor algebra $T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r)$ is a unital associative algebra over $\mathbb{F}$. We would like the multiplication on the factor algebra $T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r)/I(\mathfrak{L})$ to satisfy the following relations:

$$\ell_{xy} = \ell_x \cdot \ell_y - \ell_y \cdot \ell_x \tag{4}$$

$$r_{xy} = \ell_x \cdot r_y - r_y \cdot \ell_x \tag{5}$$

$$r_x \cdot (\ell_y + r_y) = 0 \tag{6}$$

for all elements $x, y \in \mathfrak{L}$, where $\cdot$ denotes the usual multiplication in the tensor algebra $T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r)$, or equivalently, the two-sided ideal $I(\mathfrak{L})$ of $T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r)$ is defined as the two-sided ideal generated by

$$\{\ell_x \cdot \ell_y - \ell_y \cdot \ell_x - \ell_{xy}\,,\, \ell_x \cdot r_y - r_y \cdot \ell_x - r_{xy}\,, r_x \cdot (\ell_y + r_y) \mid x, y \in \mathfrak{L}\}\,.$$

Then $\mathrm{UL}(\mathfrak{L}) := T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r)/I(\mathfrak{L})$ is called the *universal enveloping algebra* of $\mathfrak{L}$.

Remark 2.2. Note that the relations (4), (5), and (6) correspond to the identities (1), (2), and (3), respectively.

It follows from Remark 2.2 that every $\mathfrak{L}$-bimodule is a unital left $\mathrm{UL}(\mathfrak{L})$-module and vice versa (see [17, Theorem 2.3] for the analogue for right Leibniz algebras). In other words, we have an equivalence of categories:

$$\mathbf{Mod}^{\mathrm{bi}}(\mathfrak{L}) \cong \mathrm{UL}(\mathfrak{L}) - \mathbf{Mod}\,,$$

where $\mathbf{Mod}^{\mathrm{bi}}(\mathfrak{L})$ denotes the category of $\mathfrak{L}$-bimodules and $\mathrm{UL}(\mathfrak{L}) - \mathbf{Mod}$ denotes the category of unital left $\mathrm{UL}(\mathfrak{L})$-modules.

For right Leibniz algebras the following result is due to Loday and Pirashvili [17, Proposition 2.5]:

Proposition 2.3. *Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$. Then the following statements hold:*

(a) The function $d_0 : \mathrm{UL}(\mathfrak{L}) \to \mathrm{U}(\mathfrak{L}_{\mathrm{Lie}})$ defined by

$$d_0(1) := 1, \quad d_0(\ell_x) := \overline{x} \quad \textit{and} \quad d_0(r_x) := 0$$

is a homomorphism of unital associative $\mathbb{F}$-algebras.

(b) The function $d_1 : \mathrm{UL}(\mathfrak{L}) \to \mathrm{U}(\mathfrak{L}_{\mathrm{Lie}})$ defined by

$$d_1(1) := 1, \quad d_1(\ell_x) := \overline{x} \quad \textit{and} \quad d_1(r_x) := -\overline{x}$$

is a homomorphism of unital associative $\mathbb{F}$-algebras.

(c) *The function* $s_0 : \mathrm{U}(\mathfrak{L}_{\mathrm{Lie}}) \to \mathrm{UL}(\mathfrak{L})$ *defined by* $s_0(1) := 1$ *and* $s_0(\overline{x}) := \ell_x$ *is a homomorphism of unital associative* $\mathbb{F}$*-algebras.*

(d) $d_0 \circ s_0 = \mathrm{id}_{\mathrm{U}(\mathfrak{L}_{\mathrm{Lie}})}$ *and* $d_1 \circ s_0 = \mathrm{id}_{\mathrm{U}(\mathfrak{L}_{\mathrm{Lie}})}$*. In particular,* d_0 *and* d_1 *are surjective as well as* s_0 *is injective.*

(e) $\mathrm{Ker}(d_0)\,\mathrm{Ker}(d_1) = 0$.

Proof. In this proof we let $I_0(\mathfrak{L}_{\mathrm{Lie}})$ denote the ideal in the tensor algebra $T(\mathfrak{L}_{\mathrm{Lie}})$ generated by $\{x\otimes y - y\otimes x - xy \mid x, y \in \mathfrak{L}_{\mathrm{Lie}}\}$ and set $\overline{t} := t + I_0(\mathfrak{L}_{\mathrm{Lie}})$ for every element $t \in T(\mathfrak{L}_{\mathrm{Lie}})$. In particular, we have that $\mathrm{U}(\mathfrak{L}_{\mathrm{Lie}}) = T(\mathfrak{L}_{\mathrm{Lie}})/I_0(\mathfrak{L}_{\mathrm{Lie}})$ whose multiplication is denoted by $\odot$.

(a): Since d_0 is defined on generators of the algebra $\mathrm{UL}(\mathfrak{L})$, we only need to show that d_0 is well-defined, i.e., $d_0[I(\mathfrak{L})] = 0$. Namely, we have that

$$\begin{aligned} d_0(\ell_x\cdot\ell_y - \ell_y\cdot\ell_x - \ell_{xy}) &= d_0(\ell_x)\odot d_0(\ell_y) - d_0(\ell_y)\odot d_0(\ell_x) - d_0(\ell_{xy}) \\ &= \overline{x}\odot\overline{y} - \overline{y}\odot\overline{x} - \overline{xy} \\ &= \overline{x\otimes y - y\otimes x - xy} = 0\,, \end{aligned}$$

$$\begin{aligned} d_0(\ell_x\cdot r_y - r_y\cdot\ell_x - r_{xy}) &= d_0(\ell_x)\odot d_0(r_y) - d_0(r_y)\odot d_0(\ell_x) - d_0(r_{xy}) \\ &= \overline{x}\odot 0 - 0\odot\overline{x} - 0 = 0\,, \end{aligned}$$

and

$$\begin{aligned} d_0(r_x\cdot(\ell_y + r_y)) &= d_0(r_x)\odot[d_0(\ell_y) + d_0(r_y)] \\ &= 0\odot(\overline{y} + 0) = 0\,. \end{aligned}$$

(b): As in part (a), we only need to show that $d_1[I(\mathfrak{L})] = 0$. For the relation (4) the proof is exactly the same as in part (a). For the other two relations we obtain that

$$\begin{aligned} d_1(\ell_x\cdot r_y - r_y\cdot\ell_x - r_{xy}) &= d_1(\ell_x)\odot d_1(r_y) - d_1(r_y)\odot d_1(\ell_x) - d_1(r_{xy}) \\ &= \overline{x}\odot(-\overline{y}) - (-\overline{y})\odot\overline{x} - (-\overline{xy}) \\ &= -(\overline{x}\odot\overline{y}) + \overline{y}\odot\overline{x} + \overline{xy} \\ &= -\,\overline{x\otimes y - y\otimes x - xy} = 0\,, \end{aligned}$$

and

$$\begin{aligned} d_1(r_x\cdot(\ell_y + r_y)) &= d_1(r_x)\odot[d_1(\ell_y) + d_1(r_y)] \\ &= -\,\overline{x}\odot[\overline{y} + (-\overline{y})] = 0\,. \end{aligned}$$

(c): Again, we have to show that s_0 is well-defined, i.e., that $\ell_x = 0$ for every element $x \in \mathrm{Leib}(\mathfrak{L})$, or equivalently, $\ell_{x^2} = 0$ in $\mathrm{UL}(\mathfrak{L})$ for every element $x \in \mathfrak{L}$. But the latter is an immediate consequence of relation (4) in $\mathrm{UL}(\mathfrak{L})$ as

$$\ell_{x^2} = \ell_x\cdot\ell_x - \ell_x\cdot\ell_x = 0$$

for every element $x \in \mathfrak{L}$.

(d): For every element $x \in \mathfrak{L}$ we have that

$$(d_0 \circ s_0)(1) = d_0(s_0(1)) = d_0(1) = 1$$

as well as

$$(d_0 \circ s_0)(\overline{x}) = d_0(s_0(\overline{x})) = d_0(\ell_x) = \overline{x}\,,$$

and a similar argument proves the second identity.

(e): As an ideal, $\mathrm{Ker}(d_0)$ is generated by $\{r_x \mid x \in \mathfrak{L}\}$ and $\mathrm{Ker}(d_1)$ is generated by $\{\ell_y + r_y \mid y \in \mathfrak{L}\}$. Hence, the assertion is an immediate consequence of the relation (6) in $\mathrm{UL}(\mathfrak{L})$. ☐

It is well known that the universal enveloping algebra of a Lie algebra is a Hopf algebra, but is not known whether the universal enveloping algebra of a Leibniz algebra is a Hopf algebra, and this paper seems to indicate that this might not be the case. On the other hand, it follows from Proposition 2.3 that

$$\varepsilon_{\mathrm{Lie}} \circ d_0 = \varepsilon_{\mathrm{Lie}} \circ d_1 \colon \mathrm{UL}(\mathfrak{L}) \to \mathbb{F}$$

is an epimorphism of unital associative $\mathbb{F}$-algebras, where $\varepsilon_{\mathrm{Lie}}$ denotes the counit of $\mathrm{U}(\mathfrak{L}_{\mathrm{Lie}})$. Hence, $\mathrm{UL}(\mathfrak{L})$ is an *augmented unital associative* $\mathbb{F}$-*algebra* with *augmentation map* $\varepsilon_{\mathrm{Lie}} \circ d_0 = \varepsilon_{\mathrm{Lie}} \circ d_1$.

3 Weak Leibniz bimodules

We would like to make the tensor product of Leibniz bimodules into a Leibniz bimodule. Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M and N be left $\mathfrak{L}$-modules. Then, as for modules over a Lie algebra, the tensor product $M \otimes N$ (over the ground field $\mathbb{F}$) is a left $\mathfrak{L}$-module endowed with a left action

$$x \cdot (m \otimes n) := (x \cdot m) \otimes n + m \otimes (x \cdot n)$$

for all elements $x \in \mathfrak{L}$, $m \in M$, $n \in N$.

This motivates the following definition of a left and right action on the tensor product of Leibniz bimodules. For $\mathfrak{L}$-bimodules M and N we define the following left and right actions of a left Leibniz algebra $\mathfrak{L}$ on the vector space $M \otimes N$:

$$x \cdot (m \otimes n) := (x \cdot m) \otimes n + m \otimes (x \cdot n)$$

and

$$(m \otimes n) \cdot x := (m \cdot x) \otimes n + m \otimes (n \cdot x)$$

for all elements $x \in \mathfrak{L}$, $m \in M$, $n \in N$. Then, maybe a bit surprisingly, we have the following result:

Proposition 3.1. *Let $\mathfrak{L}$ be a left Leibniz algebra, and let M and N be $\mathfrak{L}$-bimodules. Then $M \otimes N$ satisfies* (LLM) *and* (LML). *Furthermore,* (MLL) *holds if, and only if,*

$$(x \cdot m + m \cdot x) \otimes (n \cdot y) + (m \cdot y) \otimes (x \cdot n + n \cdot x) = 0 \tag{7}$$

holds for all elements $x, y \in \mathfrak{L}$, $m \in M$, and $n \in N$. In particular, if M and N are both symmetric or both anti-symmetric, then $M \otimes N$ is a symmetric or anti-symmetric $\mathfrak{L}$-bimodule, respectively.

Proof. As for the tensor product of Lie modules, $M \otimes N$ satisfies (LLM). Next, let us verify that (LML) holds for $M \otimes N$. Namely, as (LML) is satisfied for M and N, we have that

$$\begin{aligned}
(m \otimes n) \cdot (xy) &= [m \cdot (xy) \otimes n] + m \otimes [n \cdot (xy)] \\
&= [x \cdot (m \cdot y)] \otimes n - [(x \cdot m) \cdot y] \otimes n \\
&\quad + m \otimes [x \cdot (n \cdot y)] - m \otimes [(x \cdot n) \cdot y] \\
&= [x \cdot (m \cdot y)] \otimes n + (m \cdot y) \otimes (x \cdot n) \\
&\quad + (x \cdot m) \otimes (n \cdot y) + m \otimes [x \cdot (n \cdot y)] \\
&\quad - [(x \cdot m) \cdot y] \otimes n - (x \cdot m) \otimes (n \cdot y) \\
&\quad - (m \cdot y) \otimes (x \cdot n) - m \otimes [(x \cdot n) \cdot y] \\
&= x \cdot [(m \cdot y) \otimes n + m \otimes (n \cdot y)] \\
&\quad - [(x \cdot m) \otimes n + m \otimes (x \cdot n)] \cdot y \\
&= x \cdot [(m \otimes n) \cdot y] - [x \cdot (m \otimes n)] \cdot y
\end{aligned}$$

for all elements $x, y \in \mathfrak{L}$, $m \in M$, and $n \in N$.

Finally, instead of (MLL), we verify (ZD) for the tensor product $M \otimes N$ (see Remark 2.1). Because (ZD) holds for M and N, we obtain that

$$\begin{aligned}
[x \cdot (m \otimes n) + (m \otimes n) \cdot x] \cdot y &= [(x \cdot m) \otimes n) + m \otimes (x \cdot n) \\
&\quad + (m \cdot x) \otimes n + m \otimes (n \cdot x)] \cdot y \\
&= [(x \cdot m) \cdot y] \otimes n + (x \cdot m) \otimes (n \cdot y) \\
&\quad + (m \cdot y) \otimes (x \cdot n) + m \otimes [(x \cdot n) \cdot y] \\
&\quad + [(m \cdot x) \cdot y] \otimes n + (m \cdot x) \otimes (n \cdot y) \\
&\quad + (m \cdot y) \otimes (n \cdot x) + m \otimes [(n \cdot x) \cdot y] \\
&= [\underbrace{(x \cdot m + m \cdot x) \cdot y}_{=0}] \otimes n + (x \cdot m + m \cdot x) \otimes (n \cdot y) \\
&\quad + (m \cdot y) \otimes (x \cdot n + n \cdot x) + m \otimes [\underbrace{(x \cdot n + n \cdot x) \cdot y}_{=0}] \\
&= (x \cdot m + m \cdot x) \otimes (n \cdot y) + (m \cdot y) \otimes (x \cdot n + n \cdot x)
\end{aligned}$$

for all elements $x, y \in \mathfrak{L}$, $m \in M$, and $n \in N$. This shows that (ZD), or equivalently (MLL), holds exactly when

$$(x \cdot m + m \cdot x) \otimes (n \cdot y) + (m \cdot y) \otimes (x \cdot n + n \cdot x) = 0$$

for all elements $x, y \in \mathfrak{L}$, $m \in M$, and $n \in N$, as asserted. Moreover, the last statement can be read off from this identity. ☐

Remark 3.2. It should be noted that in the above proof of (LLM) for the tensor product of Leibniz bimodules, we only used that (LLM) is satisfied for each of the factors. Similarly, the proof of (LML) for the tensor product of Leibniz bimodules only uses that (LML) is satisfied for each of the factors.

In the light of Proposition 3.1 and Remark 3.2, we say that a left Leibniz module with a right action satisfying (LML) is a *weak Leibniz bimodule*. Then the proof of Proposition 3.1 shows that the tensor product of weak Leibniz modules is again a weak Leibniz bimodule:

Proposition 3.3. *Let $\mathfrak{L}$ be a Leibniz algebra. If M and N are weak $\mathfrak{L}$-bimodules, then $M \otimes N$ is a weak $\mathfrak{L}$-bimodule.*

If we use the same definition of (anti-)symmetry for weak Leibniz bimodules as for ordinary Leibniz bimodules, then the following result is an immediate consequence of identity (ZD) in Remark 2.1:

Proposition 3.4. *Let $\mathfrak{L}$ be a Leibniz algebra. Then the following statements hold:*

(a) Every symmetric weak $\mathfrak{L}$-bimodule is an $\mathfrak{L}$-bimodule.

(b) Every anti-symmetric weak $\mathfrak{L}$-bimodule is an $\mathfrak{L}$-bimodule.

The next result is an immediate consequence of [8, Theorem 3.14] and Proposition 3.4:

Corollary 3.5. *Let $\mathfrak{L}$ be a Leibniz algebra. Then an irreducible weak $\mathfrak{L}$-bimodule is an $\mathfrak{L}$-bimodule if, and only if, it is symmetric or anti-symmetric.*

Remark 3.6. As in Proposition 3.1, we obtain from Proposition 3.4 that the tensor product of two symmetric or two anti-symmetric weak Leibniz bimodules is a symmetric or an anti-symmetric Leibniz bimodule, respectively. On the other hand, the tensor product of a symmetric Leibniz bimodule and an anti-symmetric Leibniz bimodule (in any order) is in general only a weak Leibniz bimodule (see Example 3.10 below). In particular, this shows that the tensor product of two ordinary Leibniz bimodules is not always an ordinary Leibniz bimodule.

Let $J(\mathfrak{L})$ denote the two-sided ideal of the tensor algebra $T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r)$ generated by

$$\{\ell_x \cdot \ell_y - \ell_y \cdot \ell_x - \ell_{xy}\,,\ \ell_x \cdot r_y - r_y \cdot \ell_x - r_{xy} \mid x, y \in \mathfrak{L}\}\,.$$

Then we say that $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) := T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r)/J(\mathfrak{L})$ is the *weak universal enveloping algebra* of $\mathfrak{L}$.

Let $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ denote the category of weak $\mathfrak{L}$-bimodules for a Leibniz algebra $\mathfrak{L}$, and let $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) - \mathbf{Mod}$ denote the category of unital left $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$-modules. It follows from Remark 2.2 that every weak $\mathfrak{L}$-bimodule is a unital left $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$-module and vice versa, and thus we have an equivalence of categories:

$$\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L}) \cong \mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) - \mathbf{Mod}\,.$$

As a consequence, we obtain that the category of weak Leibniz bimodules is an $\mathbb{F}$-linear abelian category (see [19, Chapter VIII] and [5, Definition 1.2.2]) as was already observed in [15, Section 5] for the category of bimodules over a right Leibniz algebra:

Proposition 3.7. *Let $\mathfrak{L}$ be a Leibniz algebra over a field $\mathbb{F}$. Then $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is an $\mathbb{F}$-linear abelian category.*

Next, we briefly discuss the irreducible objects of $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$. Recall that every irreducible $\mathfrak{L}$-bimodule is either symmetric or anti-symmetric (see [8, Theorem 3.14]). The main ingredient in the proof of this result is the anti-symmetric kernel M_0 of an $\mathfrak{L}$-bimodule M which is an $\mathfrak{L}$-subbimodule. Consequently, either $M_0 = 0$, i.e., M is symmetric, or $M = M_0$, i.e., M is anti-symmetric. In particular, every irreducible $\mathfrak{L}$-bimodule arises from an irreducible $\mathfrak{L}_{\mathrm{Lie}}$-module by symmetrization or anti-symmetrization (see [8, Proposition 3.15]), and therefore the classification of the irreducible $\mathfrak{L}$-bimodules reduces to the classification of the irreducible modules over its canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$. For weak Leibniz bimodules the situation is more complicated as the analogue of the anti-symmetric kernel is not necessarily anti-symmetric, i.e., its right action is not necessarily trivial (see Example 3.10 below). Of course, the reason for this is that weak Leibniz bimodules do not necessarily satisfy the identity (ZD). In fact, in general it is not clear to us whether the anti-symmetric kernel of a weak Leibniz bimodule is invariant under the right action, but the proof of [8, Proposition 3.12] shows that it is still invariant under the left action.

As a replacement for the anti-symmetric kernel of a weak $\mathfrak{L}$-bimodule M we consider

$$M\mathfrak{L} := \langle m \cdot x \mid m \in M\,,\, x \in \mathfrak{L} \rangle_{\mathbb{F}}\,,$$

which indeed is an $\mathfrak{L}$-subbimodule of M. Namely, it follows from (LML) that $M\mathfrak{L}$ is invariant under the left $\mathfrak{L}$-action and by definition $M\mathfrak{L}$ is invariant under the right $\mathfrak{L}$-action. Similarly, the space of right $\mathfrak{L}$-invariants

$$M^{\mathfrak{L}} := \{m \in M \mid \forall\, x \in \mathfrak{L} \,:\, m \cdot x = 0\}$$

is an $\mathfrak{L}$-subbimodule of M. Namely, it again follows from (LML) that $M^{\mathfrak{L}}$ is invariant under the left $\mathfrak{L}$-action and by definition $M^{\mathfrak{L}}$ is invariant under the right $\mathfrak{L}$-action.

Lemma 3.8. *Let $\mathfrak{L}$ be a Leibniz algebra, and let M be a weak $\mathfrak{L}$-bimodule. Then $M\mathfrak{L}$ and $M^{\mathfrak{L}}$ are $\mathfrak{L}$-subbimodules of M.*

From Lemma 3.8 we immediately obtain the following weak analogue of Theorem 3.14 in [8]:

Proposition 3.9. *Let $\mathfrak{L}$ be a Leibniz algebra, and let M be an irreducible weak $\mathfrak{L}$-bimodule. Then either M is anti-symmetric or $M = M\mathfrak{L}$ and $M^{\mathfrak{L}} = 0$.*

Of course, Proposition 3.9 is much weaker than its analogue for Leibniz bimodules. In the next example we obtain a complete classification of the isomorphism classes of the irreducible weak Leibniz bimodules over an algebraically closed ground field. This is possible because the weak universal enveloping algebra of the one-dimensional Lie algebra is commutative, and thus there is a geometric correspondence between irreducible modules and points in affine space.

Example 3.10. Let $\mathfrak{e} := \mathbb{F}e$ be the one-dimensional Lie algebra over a field $\mathbb{F}$. Observe that the weak universal enveloping algebra $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{e})$ is just a polynomial algebra in two variables. Namely, if we set $x := \ell_e$ and $y := r_e$, then we have that

$$\mathrm{UL}_{\mathrm{weak}}(\mathfrak{e}) = \mathbb{F}\{x, y\}/\langle xy - yx\rangle = \mathbb{F}[x, y]\,,$$

where $\mathbb{F}\{x, y\}$ denotes the free associative $\mathbb{F}$-algebra in the variables x and y, $\mathbb{F}[x, y]$ denotes the polynomial algebra in two commuting variables over $\mathbb{F}$, and $\langle xy - yx\rangle$ denotes the ideal generated by the element $xy - yx$.

Now assume that $\mathbb{F}$ is algebraically closed. Then the weak $\mathfrak{e}$-bimodules correspond to (left) $\mathbb{F}[x, y]$-modules, and the isomorphism classes of the irreducible (left) $\mathbb{F}[x, y]$-modules are in bijection with the points of the affine space $\mathbb{F}^2$ via

$$(\alpha, \beta) \mapsto \mathbb{F}[x, y]/\langle x - \alpha, y - \beta\rangle\,.$$

Note that the latter are one-dimensional $\mathbb{F}[x, y]$-modules with underlying vector space $\mathbb{F}$, where x acts via multiplication by α and y acts via multiplication by β, i.e., e acts on $\mathbb{F}$ from the left by multiplication with α and e acts on $\mathbb{F}$ from the right by multiplication with β. We denote these weak $\mathfrak{e}$-bimodules by ${}_{\alpha}F_{\beta}$. In particular, we have shown that every irreducible weak $\mathfrak{e}$-bimodule over an algebraically closed ground field is one-dimensional.

Note that ${}_{\alpha}F_{\beta}$ is symmetric exactly when $\alpha + \beta = 0$, and it is anti-symmetric exactly when $\beta = 0$. Consequently, the irreducible weak $\mathfrak{e}$-bimodule ${}_{0}F_{1}$ is neither symmetric nor anti-symmetric. Moreover, for $M := {}_{0}F_{1}$ we have that $M_0 = M$ whose right action is clearly not trivial[1].

The following example shows that the tensor product of two ordinary Leibniz bimodules is not always an ordinary Leibniz bimodule. Namely, set $M := {}_{1}F_{-1}$ and $N := {}_{1}F_{0}$. Then $M \otimes N = {}_{2}F_{-1}$ is irreducible, but neither symmetric nor anti-symmetric, and therefore $M \otimes N$ is only a weak $\mathfrak{e}$-bimodule (see Corollary 3.5).

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[1] The weak $\mathfrak{e}$-bimodule $M := {}_{0}F_{1}$ also shows that [11, Lemma 1.1] does not hold for weak Leibniz bimodules.

Remark 3.11. Note that every non-perfect Leibniz algebra admits a one-dimensional Leibniz bimodule that is neither symmetric nor anti-symmetric. Namely, let $\mathfrak{L}$ be a non-perfect Leibniz algebra. Then $\mathfrak{L}/\mathfrak{L}\mathfrak{L} \neq 0$[2], and thus $(\mathfrak{L}/\mathfrak{L}\mathfrak{L})^* \neq 0$. Now choose a non-zero linear form on $\mathfrak{L}/\mathfrak{L}\mathfrak{L}$ and lift it to a non-zero linear form λ on $\mathfrak{L}$. Then the one-dimensional weak $\mathfrak{L}$-bimodule ${}_0F_\lambda$ on which $\mathfrak{L}$ acts trivially from the left and by $\xi \cdot x := \lambda(x)\xi$ from the right is neither symmetric nor anti-symmetric.

It should also be mentioned that it is not straightforward to generalize Example 3.10 to abelian Lie algebras of dimension greater than 1 because already for a two-dimensional abelian Lie algebra $\mathfrak{a} := \mathbb{F}e \oplus \mathbb{F}f$ the weak universal enveloping algebra is not commutative. Namely, we have that

$$\mathrm{UL}_{\mathrm{weak}}(\mathfrak{a}) = \mathbb{F}\{x, y, u, v\}/(xy - yx, xu - ux, xv - vx, yu - uy, uv - vu)\,,$$

where $x := \ell_e$, $y := r_e$, $u := \ell_f$, $v := r_f$, i.e., all variables commute except for y and v.

It would be very interesting to classify all irreducible weak Leibniz bimodules up to isomorphism. We hope to come back to this problem at a later time.

Contrary to $\mathrm{UL}(\mathfrak{L})$, which is only an augmented unital associative algebra, but similar to $\mathrm{U}(\mathfrak{L}_{\mathrm{Lie}})$, $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$ is a cocommutive Hopf algebra:

Theorem 3.12. *Let $\mathfrak{L}$ be a Leibniz algebra over a field $\mathbb{F}$. Then $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$ is a cocommutative Hopf algebra with comultiplication*

$$\Delta : \mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) \to \mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) \otimes \mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$$

defined by

$$\Delta(\ell_x) := \ell_x \otimes 1 + 1 \otimes \ell_x$$

and

$$\Delta(r_x) := r_x \otimes 1 + 1 \otimes r_x\,,$$

counit $\varepsilon : \mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) \to \mathbb{F}$ defined by

$$\varepsilon(\ell_x) := 0 =: \varepsilon(r_x)\,,$$

and antipode $S : \mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) \to \mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$ defined by

$$S(\ell_x) := -\ell_x \quad \textit{and} \quad S(r_x) := -r_x\,.$$

Proof. According to [14, Theorem III.2.4 and Example 3 on p. 56], the tensor algebra $T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r)$ is a cocommutative Hopf algebra with comultiplication, counit, and antipode defined as above. In order to enable us to lift these algebra homomorphisms to the factor algebra $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$, we must show that $J(\mathfrak{L})$ is a coideal of $T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r)$ (see [14,

---

[2] Here $\mathfrak{L}\mathfrak{L} := \langle xy \mid x, y \in \mathfrak{L}\rangle_{\mathbb{F}}$ is the *derived subalgebra* of the Leibniz algebra $\mathfrak{L}$, and $\mathfrak{L}$ is called *perfect* if $\mathfrak{L} = \mathfrak{L}\mathfrak{L}$.

Definition III.1.5]) such that $S[J(\mathfrak{L})] \subseteq J(\mathfrak{L})$. For the counit by definition we have that $\varepsilon(\ell_x) = 0 = \varepsilon(r_x)$ for every $x \in \mathfrak{L}$. Moreover, we obtain that

$$\begin{aligned}
\Delta(\ell_x \cdot r_y - r_y \cdot \ell_x - r_{xy}) &= \Delta(\ell_x) \cdot \Delta(r_y) - \Delta(r_y) \cdot \Delta(\ell_x) - \Delta(r_{xy}) \\
&= (\ell_x \otimes 1 + 1 \otimes \ell_x) \cdot (r_y \otimes 1 + 1 \otimes r_y) \\
&\quad -(r_y \otimes 1 + 1 \otimes r_y) \cdot (\ell_x \otimes 1 + 1 \otimes \ell_x) \\
&\quad -(r_{xy} \otimes 1 + 1 \otimes r_{xy}) \\
&= (\ell_x \cdot r_y) \otimes 1 + \ell_x \otimes r_y + r_y \otimes \ell_x + 1 \otimes (\ell_x \cdot r_y) \\
&\quad -(r_y \cdot \ell_x) \otimes 1 - r_y \otimes \ell_x - \ell_x \otimes r_y - 1 \otimes (r_y \cdot \ell_x) \\
&\quad -(r_{xy} \otimes 1 + 1 \otimes r_{xy}) \\
&= (\ell_x \cdot r_y - r_y \cdot \ell_x - r_{xy}) \otimes 1 + 1 \otimes (\ell_x \cdot r_y - r_y \cdot \ell_x - r_{xy}) \\
&\in J(\mathfrak{L}) \otimes T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r) + T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r) \otimes J(\mathfrak{L})
\end{aligned}$$

for all elements $x, y \in \mathfrak{L}$. Similarly, we can show (as for universal enveloping algebras of Lie algebras) that

$$\Delta(\ell_x \cdot \ell_y - \ell_y \cdot \ell_x - \ell_{xy}) \in J(\mathfrak{L}) \otimes T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r) + T(\mathfrak{L}^\ell \oplus \mathfrak{L}^r) \otimes J(\mathfrak{L})$$

for all elements $x, y \in \mathfrak{L}$.

Finally, we have that

$$\begin{aligned}
S(\ell_x \cdot r_y - r_y \cdot \ell_x - r_{xy}) &= S(r_y) \cdot S(\ell_x) - S(\ell_x) \cdot S(r_y) - S(r_{xy}) \\
&= (-r_y) \cdot (-\ell_x) - (-\ell_x) \cdot (-r_y) - (-r_{xy}) \\
&= r_y \cdot \ell_x - \ell_x \cdot r_y + r_{xy} \in J(\mathfrak{L})
\end{aligned}$$

for all elements $x, y \in \mathfrak{L}$, and, similarly, we obtain that

$$S(\ell_x \cdot \ell_y - \ell_y \cdot \ell_x - \ell_{xy}) \in J(\mathfrak{L})$$

for all elements $x, y \in \mathfrak{L}$. □

Remark 3.13. As for the universal enveloping algebra of a Lie algebra, the weak universal enveloping algebra of a Leibniz algebra is generated by *primitive elements*. Note that therefore it follows from [14, Proposition III.2.6 and (3.3)] that the counit and antipode necessarily must be defined as in Theorem 3.12.

By virtue of the Cartier-Gabriel-Kostant theorem (see [5, Theorem 5.10.2]), the weak universal enveloping algebra $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$ of a Leibniz algebra $\mathfrak{L}$ over an algebraically closed field of characteristic zero is the universal enveloping algebra of the Lie algebra of its primitive elements, $\mathrm{Prim}(\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}))$. (Note that the unity is the only group-like element of $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$ because the latter is generated by primitive elements.)

It would be very useful to have an explicit description of $\mathrm{Prim}(\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}))$ in terms of a Lie algebra naturally associated with $\mathfrak{L}$. Clearly, one such candidate would be the direct sum $\mathfrak{L}_{\mathrm{Lie}} \oplus \mathfrak{L}_{\mathrm{Lie}}$ of two copies of the canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ associated with $\mathfrak{L}$. But $\mathrm{Prim}(\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}))$ is not always isomorphic to $\mathfrak{L}_{\mathrm{Lie}} \oplus \mathfrak{L}_{\mathrm{Lie}}$ as the following example shows:

Example 3.14. Let $\mathfrak{A} = \mathbb{F}h \oplus \mathbb{F}e$ be the solvable left Leibniz algebra over a field $\mathbb{F}$ with the multiplication $he = e$ and $hh = eh = ee = 0$ (see [8, Example 2.3]). If we set $x := \ell_h$, $y := r_h$, and $z := r_e$, then we have that

$$\mathrm{UL}_{\mathrm{weak}}(\mathfrak{A}) = \mathbb{F}\{x, y, z\}/\langle xy - yx, [x, z] - z\rangle \,,$$

where $\mathbb{F}\{x, y, z\}$ denotes the free associative $\mathbb{F}$-algebra in the variables x, y, z, where $[x, z] := xz - zx$ is the commutator of x and z, and where $\langle X\rangle$ denotes the two-sided ideal of $\mathbb{F}\{x, y, z\}$ that is generated by the set X.

Then for degree reasons we have that $\{x + J(\mathfrak{A}), y + J(\mathfrak{A}), z + J(\mathfrak{A})\}$ is a basis of the vector space $[\mathfrak{A}^\ell \oplus \mathfrak{A}^r + J(\mathfrak{A})]/J(\mathfrak{A})$. (Note that $he = e$, $eh = 0$, and the relation (4) imply $\ell_e = 0$.) Consequently, we obtain that

$$\dim_{\mathbb{F}}(\mathfrak{A}_{\mathrm{Lie}} \oplus \mathfrak{A}_{\mathrm{Lie}}) = 2 < \dim_{\mathbb{F}}[\mathfrak{A}^\ell \oplus \mathfrak{A}^r + J(\mathfrak{A})]/J(\mathfrak{A}) \le \dim_{\mathbb{F}} \mathrm{Prim}(\mathrm{UL}_{\mathrm{weak}}(\mathfrak{A}))\,,$$

which contradicts $\mathrm{Prim}(\mathrm{UL}_{\mathrm{weak}}(\mathfrak{A})) \cong \mathfrak{A}_{\mathrm{Lie}} \oplus \mathfrak{A}_{\mathrm{Lie}}$.

From the inclusion $J(\mathfrak{L}) \subseteq I(\mathfrak{L})$ we obtain that there is an epimorphism

$$\omega : \mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) \to \mathrm{UL}(\mathfrak{L})$$

of unital associative algebras. Let $\varepsilon_{\mathrm{Lie}} : \mathrm{U}(\mathfrak{L}_{\mathrm{Lie}}) \to \mathbb{F}$ denote the counit of $\mathrm{U}(\mathfrak{L}_{\mathrm{Lie}})$. Then we have the following composition of algebra homomorphisms:

$$\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) \overset{\omega}{\to} \mathrm{UL}(\mathfrak{L}) \overset{d_0}{\to} \mathrm{U}(\mathfrak{L}_{\mathrm{Lie}}) \overset{\varepsilon_{\mathrm{Lie}}}{\to} \mathbb{F}\,.$$

In particular, $\varepsilon = \varepsilon_{\mathrm{Lie}} \circ d_0 \circ \omega$ is the counit of $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$. Similarly, we have

$$\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) \overset{\omega}{\to} \mathrm{UL}(\mathfrak{L}) \overset{d_1}{\to} \mathrm{U}(\mathfrak{L}_{\mathrm{Lie}}) \overset{\varepsilon_{\mathrm{Lie}}}{\to} \mathbb{F}$$

and $\varepsilon = \varepsilon_{\mathrm{Lie}} \circ d_1 \circ \omega$ is the counit of $\mathrm{UL}(\mathfrak{L})$ (see Proposition 2.3 for the definitions of the algebra homomorphisms d_0 and d_1).

We obtain from Theorem 3.12 and [14, Proposition III.5.1] that $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is a symmetric monoidal category (see [19, Section VII.1, pp. 162/163, Section XI.1, pp. 252/253], [14, Definition XI.2.1], and [5, Definitions 8.1.1, 8.1.2, and 8.1.12]):

Theorem 3.15. *Let $\mathfrak{L}$ be a Leibniz algebra. Then $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is a symmetric monoidal category.*

In the following we will sketch a more down-to-earth proof of Theorem 3.15 that is neither using Theorem 3.12 nor [14, Proposition III.5.1], but instead relies on the well-known fact that the category **Vec** of vector spaces is a symmetric monoidal category, i.e., we employ the associativity and commutativity constraints as well as the left and right unit constraints that make **Vec** into a symmetric monoidal category. In particular, then the pentagon, triangle, and hexagon axioms are clearly satisfied, and we only need to verify that the associativity and commutativity constraints as well as the left and right unit constraints of **Vec** are homomorphisms of weak $\mathfrak{L}$-bimodules.

Firstly, we show that for any three weak $\mathfrak{L}$-bimodules L, M, and N the canonical isomorphism

$$\alpha_{L,M,N} : (L \otimes M) \otimes N \to L \otimes (M \otimes N)\,,\ (l \otimes m) \otimes n \mapsto l \otimes (m \otimes n)$$

is a homomorphism of weak $\mathfrak{L}$-bimodules, i.e., $\alpha_{L,M,N}$ is compatible with the left and right $\mathfrak{L}$-actions. Here we only verify the compatibility with the right $\mathfrak{L}$-action and leave the completely analogous proof for the compatibility with the left $\mathfrak{L}$-action to the interested reader. We have that

$$\begin{aligned}
&\alpha_{L,M,N}([(l \otimes m) \otimes n] \cdot x)\\
&\qquad = \alpha_{L,M,N}([(l \otimes m) \cdot x] \otimes n + (l \otimes m) \otimes (n \cdot x))\\
&\qquad = \alpha_{L,M,N}([(l \cdot x) \otimes m] \otimes n + [l \otimes (m \cdot x)] \otimes n + (l \otimes m) \otimes (n \cdot x))\\
&\qquad = (l \cdot x) \otimes (m \otimes n) + l \otimes [(m \cdot x) \otimes n] + l \otimes [m \otimes (n \cdot x)]\\
&\qquad = (l \cdot x) \otimes (m \otimes n) + l \otimes [(m \otimes n) \cdot x]\\
&\qquad = [l \otimes (m \otimes n)] \cdot x\\
&\qquad = \alpha_{L,M,N}((l \otimes m) \otimes n) \cdot x
\end{aligned}$$

for all elements $l \in L$, $m \in M$, $n \in N$, and $x \in \mathfrak{L}$.

Secondly, we show that for any two weak $\mathfrak{L}$-bimodules M and N the flip

$$\gamma_{M,N} : M \otimes N \to N \otimes M\,,\ m \otimes n \mapsto n \otimes m$$

is a homomorphism of weak $\mathfrak{L}$-bimodules. Here we again only verify that $\gamma_{M,N}$ is compatible with the right $\mathfrak{L}$-action and leave the proof for the compatibility of $\gamma_{M,N}$ with the left $\mathfrak{L}$-action to the interested reader. We have that

$$\begin{aligned}
\gamma_{M,N}((m \otimes n) \cdot x) &= \gamma_{M,N}((m \cdot x) \otimes n + m \otimes (n \cdot x))\\
&= n \otimes (m \cdot x) + (n \cdot x) \otimes m\\
&= (n \cdot x) \otimes m + n \otimes (m \cdot x)\\
&= (n \otimes m) \cdot x\\
&= \gamma_{M,N}(m \otimes n) \cdot x
\end{aligned}$$

for all elements $m \in M$, $n \in N$, and $x \in \mathfrak{L}$.

Finally, we show that for every weak $\mathfrak{L}$-bimodule M the left unit

$$\lambda_M : \mathbb{F} \otimes M \to M\,,\ 1 \otimes m \mapsto m$$

and the right unit

$$\rho_M : M \otimes \mathbb{F} \to M\,,\ m \otimes 1 \mapsto m$$

are homomorphisms of weak $\mathfrak{L}$-bimodules. Here we only show that λ_M is compatible with the right $\mathfrak{L}$-action and leave the remaining proofs to the interested reader. Namely, let $x \in \mathfrak{L}$ and $m \in M$ be arbitrary. Then we have that

$$\begin{aligned} \lambda_M[(\beta \otimes m) \cdot x] &= \lambda_M[\beta \otimes (m \cdot x)] = \beta(m \cdot x) \\ &= (\beta m) \cdot x = \lambda_M(\beta \otimes m) \cdot x \end{aligned}$$

for all elements $\beta \in \mathbb{F}$, $m \in M$, and $x \in \mathfrak{L}$.

As to be expected, the tensor product of weak Leibniz bimodules is compatible with direct sums:

Proposition 3.16. *Let $\mathfrak{L}$ be a Leibniz algebra, and let L, M, N be weak $\mathfrak{L}$-bimodules. Then there are natural isomorphisms*

$$L \otimes (M \oplus N) \cong (L \otimes M) \oplus (L \otimes N)$$

and

$$(L \oplus M) \otimes N \cong (L \otimes N) \oplus (M \otimes N)$$

of weak $\mathfrak{L}$-bimodules.

As for Theorem 3.15, the natural isomorphisms are again the ones in the symmetric monoidal category of **Vec**, and similar to the previous paragraph, one can verify that these are morphisms in $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$.

Clearly, the category $\mathbf{Mod}^{\mathrm{bi}}(\mathfrak{L})$ of $\mathfrak{L}$-bimodules is a full subcategory of the category $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ of weak $\mathfrak{L}$-bimodules. In fact, more is true: The forgetful (or underlying or inclusion or restriction) functor

$$U : \mathbf{Mod}^{\mathrm{bi}}(\mathfrak{L}) \to \mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$$

admits the left adjoint induction functor

$$I : \mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L}) \to \mathbf{Mod}^{\mathrm{bi}}(\mathfrak{L})$$

defined by $M \mapsto \mathrm{UL}(\mathfrak{L}) \otimes_{\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})} M$, where $\mathrm{UL}(\mathfrak{L})$ is a right $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$-module via the epimorphism $\omega : \mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) \to \mathrm{UL}(\mathfrak{L})$ of unital associative algebras. This is indeed an adjunction:

$$\mathrm{Hom}_{\mathbf{Mod}^{\mathrm{bi}}(\mathfrak{L})}(I(M), N) \cong \mathrm{Hom}_{\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})}(M, U(N))\,.$$

In fact, this natural isomorphism is the usual change-of-rings adjunction (Frobenius reciprocity or Shapiro's lemma).

Recall that a subcategory is called **reflective** if the inclusion functor admits a left adjoint (see [19, p. 91]). Thus, we have

Proposition 3.17. *Let $\mathfrak{L}$ be a Leibniz algebra. Then $\mathbf{Mod}^{\mathrm{bi}}(\mathfrak{L})$ is a reflective subcategory of $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$.*

Clearly, the inclusion functor $U : \mathbf{Mod}^{\mathrm{bi}}(\mathfrak{L}) \to \mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is fully faithful. Hence, we obtain from [19, Theorem 1, p. 90] the following result:

Proposition 3.18. *If $\mathfrak{L}$ is a Leibniz algebra, then for every $\mathfrak{L}$-bimodule M we have the natural isomorphism*

$$\mathrm{UL}(\mathfrak{L}) \otimes_{\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})} U(M) \cong M$$

of $\mathfrak{L}$-bimodules.

It would be interesting to know whether the category of weak Leibniz bimodules is the *smallest* symmetric monoidal category containing the category of Leibniz bimodules as a reflective subcategory.

Let $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ denote the category of finite-dimensional weak $\mathfrak{L}$-bimodules for a Leibniz algebra $\mathfrak{L}$ which is a full subcategory of the category $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ of all weak $\mathfrak{L}$-bimodules. If $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) - \mathbf{mod}$ denotes the category of finite-dimensional unital left $\mathrm{UL}_{\mathrm{weak}}(\mathfrak{L})$-modules, then we have the following equivalence of categories:

$$\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L}) \cong \mathrm{UL}_{\mathrm{weak}}(\mathfrak{L}) - \mathbf{mod}\,.$$

As a consequence, we obtain that the category of finite-dimensional weak Leibniz bimodules is a locally finite $\mathbb{F}$-linear abelian category (see [19, Chapter VIII] or [5, Definitions 1.8.1, 1.2.2, and 1.3.1, respectively]):

Proposition 3.19. *Let $\mathfrak{L}$ be a Leibniz algebra over a field $\mathbb{F}$. Then $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is a locally finite $\mathbb{F}$-linear abelian category.*

Remark 3.20. It follows from Example 3.10 that $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is not always finite (see condition (iv) in [5, Definition 1.8.6]).

Moreover, it is clear that $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is a full monoidal subcategory of the monoidal category $\mathbf{Mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ (see [5, Definition 2.1.4][3]). We conclude this section by showing that $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is rigid, i.e., every finite-dimensional weak $\mathfrak{L}$-bimodule has a left and a right dual (see [5, Definitions 2.10.1, 2.10.2, and 2.10.11]).

First, we need to show that the linear dual of a weak Leibniz bimodule is again a weak Leibniz bimodule. More generally, we prove that the vector space of linear transformations between weak Leibniz bimodules is a weak Leibniz bimodule.

Let $\mathfrak{L}$ be a left Leibniz algebra, and let M and N be $\mathfrak{L}$-bimodules. We define left and right actions of $\mathfrak{L}$ on the vector space $\mathrm{Hom}_{\mathbb{F}}(M, N)$ of linear transformations from M to

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[3] Note that here we use [5, Definition 2.2.8] (or [19, pp. 162/163] resp. [14, Definition XI.2.1]) instead of [5, Definition 2.1.1] for the definition of a monoidal category. Then a *full monoidal subcategory* of a monoidal category $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ is a sextuple $(\mathcal{D}, \otimes, 1, \alpha, \lambda, \rho)$ such that $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ that contains the unit 1 and is closed under tensor products of objects and morphisms.

N as follows:

$$(x\cdot f)(m) := x\cdot f(m) - f(x\cdot m)$$

and

$$(f\cdot x)(m) := f(m)\cdot x - f(m\cdot x)$$

for all elements $x\in\mathfrak{L}$, $f\in\mathrm{Hom}_{\mathbb{F}}(M,N)$, and $m\in M$. Then we have the following result (cf. Proposition 3.1):

Proposition 3.21. *Let $\mathfrak{L}$ be a left Leibniz algebra over a field $\mathbb{F}$, and let M and N be $\mathfrak{L}$-bimodules. Then $\mathrm{Hom}_{\mathbb{F}}(M,N)$ is a weak $\mathfrak{L}$-bimodule. In particular, if M and N are both symmetric or both anti-symmetric, then $\mathrm{Hom}_{\mathbb{F}}(M,N)$ is a symmetric or anti-symmetric $\mathfrak{L}$-bimodule, respectively.*

Proof. As is true for Lie modules, $\mathrm{Hom}_{\mathbb{F}}(M,N)$ satisfies (LLM) (see also the first part of the proof of [11, Lemma 1.4 (b)] for the special case $M=\mathfrak{L}_{\mathrm{ad}}$).

Next, let us verify that (LML) holds. Namely, as (LML) is satisfied for M and N, we have that

$$\begin{aligned}
[f\cdot(xy)](m) &= f(m)\cdot(xy) - f[m\cdot(xy)]\\
&= x\cdot[f(m)\cdot y] - [x\cdot f(m)]\cdot y - f[x\cdot(m\cdot y)] + f[(x\cdot m)\cdot y]\\
&= x\cdot[f(m)\cdot y] - x\cdot f(m\cdot y) - f(x\cdot m)\cdot y + f[(x\cdot m)\cdot y]\\
&\quad -[x\cdot f(m)]\cdot y + f(x\cdot m)\cdot y + x\cdot f(m\cdot y) - f[x\cdot(m\cdot y)]\\
&= x\cdot(f\cdot y)(m) - (f\cdot y)(x\cdot m) - (x\cdot f)(m)\cdot y + (x\cdot f)(m\cdot y)\\
&= [x\cdot(f\cdot y)](m) - [(x\cdot f)\cdot y](m)
\end{aligned}$$

for all elements $f\in\mathrm{Hom}_{\mathbb{F}}(M,N)$, $x,y\in\mathfrak{L}$, and $m\in M$.

Finally, let us consider (ZD). Because (ZD) holds for N, we obtain that

$$\begin{aligned}
[(x\cdot f+f\cdot x)\cdot y](m) &= (x\cdot f+f\cdot x)(m)\cdot y - (x\cdot f+f\cdot x)(m\cdot y)\\
&= (x\cdot f)(m)\cdot y + (f\cdot x)(m)\cdot y - (x\cdot f)(m\cdot y) - (f\cdot x)(m\cdot y)\\
&= [x\cdot f(m)]\cdot y - f(x\cdot m)\cdot y + [f(m)\cdot x]\cdot y - f(m\cdot x)\cdot y\\
&\quad -x\cdot f(m\cdot y) + f[x\cdot(m\cdot y)] - f(m\cdot y)\cdot x + f[(m\cdot y)\cdot x]\\
&= \underbrace{[x\cdot f(m)+f(m)\cdot x]\cdot y}_{=0} - f(x\cdot m+m\cdot x)\cdot y\\
&\quad -x\cdot f(m\cdot y) - f(m\cdot y)\cdot x + f[x\cdot(m\cdot y)+(m\cdot y)\cdot x]\\
&= f[x\cdot(m\cdot y)+(m\cdot y)\cdot x] - [x\cdot f(m\cdot y)+f(m\cdot y)\cdot x]\\
&\quad -f(x\cdot m+m\cdot x)\cdot y
\end{aligned}$$

for all elements $f\in\mathrm{Hom}_{\mathbb{F}}(M,N)$, $x,y\in\mathfrak{L}$, and $m\in M$. From the last expression we conclude that $\mathrm{Hom}_{\mathbb{F}}(M,N)$ is an (anti-)symmetric $\mathfrak{L}$-bimodule if both M and N are (anti-)symmetric. $\square$

Remark 3.22. It should be noted that in the above proof of (LML), we only used that (LML) is satisfied for each of the bimodules. Similarly, the proof of (LLM) only uses that (LLM) is satisfied for each of the bimodules. Moreover, it is noteworthy that in the last part of the proof of Proposition 3.21 only the second bimodule needs to satisfy (ZD).

The proof of Proposition 3.21 shows that the vector space of linear transformations between weak Leibniz bimodules is again a weak Leibniz bimodule:

Proposition 3.23. *Let $\mathfrak{L}$ be a Leibniz algebra over a field $\mathbb{F}$. If M and N are weak $\mathfrak{L}$-bimodules, then $\mathrm{Hom}_{\mathbb{F}}(M,N)$ is a weak $\mathfrak{L}$-bimodule. In particular, the linear dual M^* of a weak $\mathfrak{L}$-bimodule is a weak $\mathfrak{L}$-bimodule.*

Let $\mathfrak{L}$ be a Leibniz algebra over a field $\mathbb{F}$, and let M be a weak $\mathfrak{L}$-bimodule. Define the contractions

$$\mathrm{ev}_M : M^* \otimes M \to \mathbb{F}\,,\ \mu \otimes m \mapsto \mu(m)$$

and

$$\mathrm{ev}'_M : M \otimes M^* \to \mathbb{F}\,,\ m \otimes \mu \mapsto \mu(m)\,.$$

If M is finite dimensional, choose a basis $\{m_1,\dots,m_d\}$ of M and the canonical dual basis $\{m_1^*,\dots,m_d^*\}$ such that $m_i^*(m_j)=\delta_{ij}$ for all $i,j\in\{1,\dots,d\}$, where δ_{ij} is the usual Kronecker delta. Then we can define the embeddings

$$\mathrm{coev}_M : \mathbb{F} \to M \otimes M^*\,,\ 1 \mapsto \sum_{i=1}^{d} m_i \otimes m_i^*$$

and

$$\mathrm{coev}'_M : \mathbb{F} \to M^* \otimes M\,,\ 1 \mapsto \sum_{i=1}^{d} m_i^* \otimes m_i$$

of $\mathbb{F}$-vector spaces. Note that

$$\mathrm{ev}_M \circ \mathrm{coev}'_M = (\dim_{\mathbb{F}} M)\cdot \mathrm{id}_{\mathbb{F}} \quad \text{and} \quad \mathrm{ev}'_M \circ \mathrm{coev}_M = (\dim_{\mathbb{F}} M)\cdot \mathrm{id}_{\mathbb{F}}\,.$$

In particular, ev_M, ev'_M are surjective and coev_M, coev'_M are injective.

Theorem 3.24. *Let $\mathfrak{L}$ be a Leibniz algebra. Then $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is a rigid symmetric monoidal category.*

Proof. It is well known that the category of finite-dimensional vector spaces is a rigid symmetric monoidal category with the contractions and embeddings defined as above (see [5, Example 2.10.12]), where the linear dual V^* of a finite-dimensional vector space V is a left and right dual of V. Consequently, it is enough to prove that ev_M, ev'_M, coev_M, and coev'_M are homomorphisms of weak $\mathfrak{L}$-bimodules for every finite-dimensional weak $\mathfrak{L}$-bimodule M.

Let M be a finite-dimensional weak $\mathfrak{L}$-bimodule. We will show that ev_M and coev'_M are compatible with the right action of $\mathfrak{L}$ and leave the remaining proofs, which are very similar, to the interested reader.

Firstly, we prove that ev_M is compatible with the right action of $\mathfrak{L}$. Since $\mathfrak{L}$ acts trivially on $\mathbb{F}$, we need to show that $\mathrm{ev}_M[(\mu\otimes m)\cdot x]=0$ for all elements $x\in\mathfrak{L}$, $\mu\in M^*$, and $m\in M$:

$$\begin{aligned}\mathrm{ev}_M[(\mu\otimes m)\cdot x] &= \mathrm{ev}_M[(\mu\cdot x)\otimes m+\mu\otimes(m\cdot x)]\\ &= (\mu\cdot x)(m)+\mu(m\cdot x)\\ &= -\mu(m\cdot x)+\mu(m\cdot x)=0\,.\end{aligned}$$

Next, we prove that coev'_M is compatible with the right action of $\mathfrak{L}$. For this we first need to show that if $m_i\cdot x=\sum_{j=1}^d\xi_{ij}m_j$, then $m_i^*\cdot x=-\sum_{j=1}^d\xi_{ji}m_j^*$ for every integer $i\in\{1,\dots,d\}$. Namely, let $m=\sum_{k=1}^d\beta_km_k$ be arbitrary. Then we have that

$$\begin{aligned}(m_i^*\cdot x)(m) &= -m_i^*(m\cdot x)\\ &= -\sum_{k=1}^d\beta_km_i^*(m_k\cdot x)\\ &= -\sum_{k,l=1}^d\beta_k\xi_{kl}m_i^*(m_l)\\ &= -\sum_{k=1}^d\beta_k\xi_{ki}\\ &= -\sum_{j,k=1}^d\beta_k\xi_{ji}m_j^*(m_k)\\ &= -\sum_{j=1}^d\xi_{ji}m_j^*(m)\\ &= -\left(\sum_{j=1}^d\xi_{ji}m_j^*\right)(m)\,,\end{aligned}$$

and therefore we obtain that

$$\begin{aligned}\sum_{i=1}^d m_i^*\otimes(m_i\cdot x) &= \sum_{i=1}^d\left[m_i^*\otimes\left(\sum_{j=1}^d\xi_{ij}m_j\right)\right]\\ &= \sum_{j=1}^d\left[\left(\sum_{i=1}^d\xi_{ij}m_i^*\right)\otimes m_j\right]\\ &= -\sum_{j=1}^d(m_j^*\cdot x)\otimes m_j\,.\end{aligned}$$

Now we can prove that coev'_M is compatible with the right action of $\mathfrak{L}$. Since $\mathfrak{L}$ acts trivially on $\mathbb{F}$, we need to show that $\mathrm{coev}'_M(\sigma) \cdot x = 0$ for all elements $x \in \mathfrak{L}$ and $\sigma \in \mathbb{F}$:

$$\begin{aligned} \mathrm{coev}'_M(\sigma) \cdot x &= \sigma \left(\sum_{i=1}^{d} m_i^* \otimes m_i \right) \cdot x \\ &= \sigma \sum_{i=1}^{d} [(m_i^* \cdot x) \otimes m_i + m_i^* \otimes (m_i \cdot x)] = 0 \,, \end{aligned}$$

as desired. □

Remark 3.25. One can also show for every finite-dimensional weak $\mathfrak{L}$-bimodule M that the natural isomorphism $M \cong M^{**}$ of vector spaces is compatible with the left and right $\mathfrak{L}$-actions, i.e., $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is *pivotal* (see [5, Definition 4.7.8]).

According to Proposition 3.19 and Theorem 3.24, for every Leibniz algebra $\mathfrak{L}$ over a field $\mathbb{F}$ we have that $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is a locally finite $\mathbb{F}$-linear abelian rigid symmetric monoidal category. Moreover, clearly, the bifunctor

$$\otimes : \mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L}) \times \mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L}) \to \mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$$

is $\mathbb{F}$-bilinear on morphisms and $\mathrm{End}_{\mathfrak{L}}(\mathbb{F}) \cong \mathbb{F}$. Hence, we obtain that $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is a *tensor category* over $\mathbb{F}$ in the sense of Etingof et al.[4] (see [5, Definition 4.1.1]), and therefore it follows from [5, Proposition 4.2.1] that the bifunctor $\otimes$ is biexact[5]. Consequently, $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is a *ring category* over $\mathbb{F}$ (see [5, Definition 4.2.3]):

Theorem 3.26. *Let $\mathfrak{L}$ be a Leibniz algebra over a field $\mathbb{F}$. Then $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ is a ring category over $\mathbb{F}$.*

In Section 5 we will study the Grothendieck ring of $\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ and a certain non-associative Grothendieck ring defined for the category $\mathbf{mod}^{\mathrm{bi}}(\mathfrak{L})$ of all finite-dimensional $\mathfrak{L}$-bimodules and compare them with the Grothendieck ring of the category $\mathbf{mod}(\mathfrak{L}_{\mathrm{Lie}})$ of finite-dimensional $\mathfrak{L}_{\mathrm{Lie}}$-modules. Before we can do this, we need to introduce a tensor product on the category of (finite-dimensional) Leibniz bimodules.

4 Truncated tensor products for Leibniz bimodules

In this section we define for $\mathfrak{L}$-bimodules M and N over a left Leibniz algebra $\mathfrak{L}$ two truncated tensor products $M \overline{\otimes} N$ and $M \underline{\otimes} N$ which both are again $\mathfrak{L}$-bimodules. According to (7) in Proposition 3.1, we must ensure that

$$(x \cdot m + m \cdot x) \otimes (n \cdot y) + (m \cdot y) \otimes (x \cdot n + n \cdot x) = 0$$

---

[4] Note that in Kassel's book a tensor category is the same as a monoidal category (see [14, Definition XI.2.1]).

[5] A bifunctor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is *biexact* if it is exact in both factors.

holds for all elements $x, y \in \mathfrak{L}$, $m \in M$, and $n \in N$.

Let $T(M,N)$ be the smallest subspace of the tensor product $M \otimes N$ that contains

$$S(M,N) := \{(x \cdot m + m \cdot x) \otimes (n \cdot y) + (m \cdot y) \otimes (x \cdot n + n \cdot x) \mid x, y \in \mathfrak{L}; m \in M, n \in N\}$$

and that is closed under the left and right $\mathfrak{L}$-actions as defined in Section 3, i.e., $T(M,N)$ is the weak $\mathfrak{L}$-bimodule of $M \otimes N$ generated by $S(M,N)$. Now define

$$M \overline{\otimes} N := (M \otimes N)/T(M,N)\,.$$

On the other hand, set

$$T_0(M,N) := M_0 \otimes N\mathfrak{L} + M\mathfrak{L} \otimes N_0\,,$$

where M_0 and N_0 are the *anti-symmetric kernels* of M and N, respectively, which both are anti-symmetric $\mathfrak{L}$-subbimodules (see [8, Proposition 3.12]), and $M\mathfrak{L}$ is the subspace of M that is spanned by $\{m \cdot x \mid m \in M, x \in \mathfrak{L}\}$, and similarly, for $N\mathfrak{L}$. Since every factor in the tensor products of $T_0(M,N)$ is an $\mathfrak{L}$-subbimodule, we obtain that $T_0(M,N)$ is a weak $\mathfrak{L}$-subbimodule of $M \otimes N$, and we define

$$M \underline{\otimes} N := (M \otimes N)/T_0(M,N)\,.$$

Remark 4.1. Note that the truncated tensor product $M \overline{\otimes} N$ even makes sense for weak $\mathfrak{L}$-bimodules M and N. On the other hand, this is not clear for the truncated tensor product $M \underline{\otimes} N$, although $M\mathfrak{L}$ and $N\mathfrak{L}$ are weak $\mathfrak{L}$-subbimodules of weak $\mathfrak{L}$-bimodules M and N, respectively (see Lemma 3.8), but this seems not to be the case for the anti-symmetric kernels M_0 and N_0, respectively (see also our discussion after Proposition 3.7).

It follows from Proposition 3.1 that the truncated tensor products of Leibniz bimodules are again Leibniz bimodules:

Proposition 4.2. *Let $\mathfrak{L}$ be a Leibniz algebra, and let M and N be $\mathfrak{L}$-bimodules. Then the truncated tensor products $M \overline{\otimes} N$ and $M \underline{\otimes} N$ are $\mathfrak{L}$-bimodules.*

Remark 4.3. Observe that factoring out $T_0(M,N) = M_0 \otimes N\mathfrak{L} + M\mathfrak{L} \otimes N_0$ is factoring out more than $T(M,N)$, because in the definition of $S(M,N)$ the two summands are linked, while in $T_0(M,N)$ these terms are not linked. Consequently, we have the inclusion $T(M,N) \subseteq T_0(M,N)$, but since we could not find any example where $T(M,N)$ is properly contained in $T_0(M,N)$, at the moment it is not clear to us whether both truncated tensor products always coincide.

It is not surprising that both truncated tensor products are commutative:

Proposition 4.4. *Let $\mathfrak{L}$ be a Leibniz algebra, and let M and N be $\mathfrak{L}$-bimodules. Then there are natural isomorphisms $M \overline{\otimes} N \cong N \overline{\otimes} M$ and $M \underline{\otimes} N \cong N \underline{\otimes} M$ of $\mathfrak{L}$-bimodules.*

Proof. Let $\gamma : M \otimes N \to N \otimes M$ be the natural isomorphism of weak $\mathfrak{L}$-bimodules defined by $\gamma(m \otimes n) := n \otimes m$ (cf. the down-to-earth proof of Theorem 3.15). It is easy to see that $\gamma[T(M, N)] = T(N, M)$. Now consider the homomorphism $\overline{\gamma} := \overline{\eta} \circ \gamma : M \otimes N \to N \overline{\otimes} M$ of weak $\mathfrak{L}$-bimodules, where $\overline{\eta} : N \otimes M \to N \overline{\otimes} M$ is the natural epimorphism. As a composition of surjective functions, $\overline{\gamma}$ is also surjective. So it remains to prove that $\mathrm{Ker}(\overline{\gamma}) = T(M, N)$ in order to establish the first assertion. But the latter follows from $\gamma[T(M, N)] = T(N, M)$.

Similar to the above proof, we observe that $\gamma[T_0(M, N)] = T_0(N, M)$, and then we consider the homomorphism $\underline{\gamma} := \underline{\eta} \circ \gamma : M \otimes N \to N \underline{\otimes} M$ of weak $\mathfrak{L}$-bimodules, where $\underline{\eta} : N \otimes M \to N \underline{\otimes} M$ is the natural epimorphism. Of course, $\underline{\gamma}$ is surjective. So it remains to prove that $\mathrm{Ker}(\underline{\gamma}) = T_0(M, N)$ which immediately follows from the fact that $\gamma[T_0(M, N)] = T_0(N, M)$. ☐

It should be noted that neither of the truncated tensor products is associative. We will come back to this later in this section (see Proposition 4.10) and in Section 5 (see Corollary 5.11). But both truncated tensor products are compatible with direct sums:

Proposition 4.5. *Let $\mathfrak{L}$ be a Leibniz algebra, and let L, M, and N be $\mathfrak{L}$-bimodules. Then there are natural isomorphisms*

$$\begin{aligned} L \overline{\otimes} (M \oplus N) &\cong (L \overline{\otimes} M) \oplus (L \overline{\otimes} N)\,, \\ L \underline{\otimes} (M \oplus N) &\cong (L \underline{\otimes} M) \oplus (L \underline{\otimes} N)\,, \\ (L \oplus M) \overline{\otimes} N &\cong (L \overline{\otimes} N) \oplus (M \overline{\otimes} N)\,, \\ (L \oplus M) \underline{\otimes} N &\cong (L \underline{\otimes} N) \oplus (M \underline{\otimes} N) \end{aligned}$$

of $\mathfrak{L}$-bimodules.

Proof. According to Proposition 4.4, it is enough to prove the first two statements. By virtue of Proposition 3.16, we have the natural isomorphism

$$\delta : L \otimes (M \oplus N) \to (L \otimes M) \oplus (L \otimes N)\,,\ l \otimes (m, n) \mapsto (l \otimes m, l \otimes n)$$

of weak $\mathfrak{L}$-bimodules.

In order to establish the first isomorphism, it remains to show that δ lifts to the corresponding truncated tensor products. For this, we will need that

$$\delta[S(L, M \oplus N)] \subseteq S(L, M) \times S(L, N)\,,$$

which in turn implies that

$$\delta[T(L, M \oplus N)] \subseteq T(L, M) \times T(L, N)\,.$$

Every element $s \in S(L, M \oplus N)$ can be written as

$$s = (x \cdot l + l \cdot x) \otimes (m, n) \cdot y + l \cdot y \otimes [x \cdot (m, n) + (m, n) \cdot x]$$

for some elements $l \in L$, $m \in M$, $n \in N$, and $x, y \in \mathfrak{L}$. Then we have that

$$\begin{aligned}\delta[(x \cdot l + l \cdot x) \otimes (m, n) \cdot y] &= \delta(x \cdot l \otimes (m \cdot y, n \cdot y) + l \cdot x \otimes (m \cdot y, n \cdot y)) \\ &= (x \cdot l \otimes m \cdot y, x \cdot l \otimes n \cdot y) + (l \cdot x \otimes m \cdot y, l \cdot x \otimes n \cdot y) \\ &= ((x \cdot l + l \cdot x) \otimes m \cdot y, (x \cdot l + l \cdot x) \otimes n \cdot y)\end{aligned}$$

and

$$\begin{aligned}&\delta(l \cdot y \otimes [x \cdot (m, n) + (m, n) \cdot x]) \\ &\quad = \delta(l \cdot y \otimes (x \cdot m, x \cdot n) + l \cdot y \otimes (m \cdot x, n \cdot x)) \\ &\quad = (l \cdot y \otimes m \cdot x, l \cdot y \otimes n \cdot x) + (l \cdot y \otimes x \cdot m, l \cdot y \otimes x \cdot n) \\ &\quad = ((l \cdot y \otimes (x \cdot m + m \cdot x), l \cdot y \otimes (x \cdot n + n \cdot x))\,,\end{aligned}$$

which implies that

$$\begin{aligned}\delta(s) &= ((x \cdot l + l \cdot x) \otimes m \cdot y + l \cdot y \otimes (x \cdot m + m \cdot x), \\ &\qquad (x \cdot l + l \cdot x) \otimes n \cdot y + l \cdot y \otimes (x \cdot n + n \cdot x)) \\ &\in S(L, M) \times S(L, N)\,.\end{aligned}$$

It follows directly from the additivity of the functors $M \mapsto M_0$ and $M \mapsto M\mathfrak{L}$ that δ lifts to the second isomorphism. ☐

In the remainder of this section we will discuss when the two truncated tensor products coincide or are non-zero. In particular, the former happens when one of the bimodules is symmetric or anti-symmetric. Recall that an $\mathfrak{L}$-bimodule M is symmetric exactly when $M_0 = 0$, and M is anti-symmetric exactly when $M\mathfrak{L} = 0$.

Theorem 4.6. *Let $\mathfrak{L}$ be a Leibniz algebra, and let M and N be $\mathfrak{L}$-bimodules. Then the following statements hold:*

(a) If M is symmetric, then

$$M \overline{\otimes} N = M \underline{\otimes} N = (M \otimes N)/(\mathfrak{L}M \otimes N_0)\,.$$

(b) If M is anti-symmetric, then

$$M \overline{\otimes} N = M \underline{\otimes} N = (M \otimes N)/(\mathfrak{L}M \otimes N\mathfrak{L})\,.$$

(c) If N is symmetric, then

$$M \overline{\otimes} N = M \underline{\otimes} N = (M \otimes N)/(M_0 \otimes \mathfrak{L}N)\,.$$

(d) If N is anti-symmetric, then

$$M \overline{\otimes} N = M \underline{\otimes} N = (M \otimes N)/(M\mathfrak{L} \otimes \mathfrak{L}N)\,.$$

Proof. Note that $X\mathfrak{L} = \mathfrak{L}X$ for a symmetric $\mathfrak{L}$-bimodule X and $Y_0 = \mathfrak{L}Y$ for an anti-symmetric $\mathfrak{L}$-bimodule Y. This in conjunction with the definition of $T_0(M,N)$ yields the statements (a) – (d) for $M\underline{\otimes}N$. By virtue of Proposition 4.4, it remains to prove that $T(M,N) = T_0(M,N)$ if M is either symmetric or anti-symmetric.

Suppose that M is symmetric. Then $M_0 = 0$, and thus $T_0(M,N) = M\mathfrak{L} \otimes N_0$ and

$$S(M,N) = \{(m \cdot y) \otimes (x \cdot n + n \cdot x) \mid x, y \in \mathfrak{L}; m \in M, n \in N\}\,.$$

By using (LML) and the argument in the proof of Proposition 3.12 in [8], we see then that $\langle S(M,N)\rangle_{\mathbb{F}}$ is a weak $\mathfrak{L}$-subbimodule of $M \otimes N$, and therefore we obtain that

$$T(M,N) = \langle S(M,N)\rangle_{\mathbb{F}} = M\mathfrak{L} \otimes N_0 = T_0(M,N).$$

The proof in the other case works very similar and is left to the interested reader. ☐

In particular, we have for two (anti-)symmetric Leibniz bimodules that both truncated tensor products coincide with the "natural" tensor product defined in Section 3:

Corollary 4.7. *Let $\mathfrak{L}$ be a Leibniz algebra, and let M and N be $\mathfrak{L}$-bimodules. Then the following statements hold:*

(a) If M and N are symmetric, then $M\overline{\otimes}N = M\underline{\otimes}N = M \otimes N$.

(b) If M and N are anti-symmetric, then $M\overline{\otimes}N = M\underline{\otimes}N = M \otimes N$.

(c) If M is symmetric and N is anti-symmetric, then

$$M\overline{\otimes}N = M\underline{\otimes}N = (M \otimes N)/(\mathfrak{L}M \otimes \mathfrak{L}N)\,.$$

(d) If M is anti-symmetric and N is symmetric, then

$$M\overline{\otimes}N = M\underline{\otimes}N = (M \otimes N)/(\mathfrak{L}M \otimes \mathfrak{L}N)\,.$$

Proof. (a) is an immediate consequence of Theorem 4.6 (a) or (c), and similarly, (b) follows from Theorem 4.6 (b) or (d). Moreover, (c) is a special case of Theorem 4.6 (a) or (d), and (d) is a special case of Theorem 4.6 (b) or (c). ☐

Recall that a *trivial Leibniz bimodule* is a Leibniz bimodule with trivial left and right actions. In particular, the ground field $\mathbb{F}$ of a Leibniz algebra $\mathfrak{L}$ with trivial left and right $\mathfrak{L}$-actions is called the *one-dimensional trivial $\mathfrak{L}$-bimodule* and will be denoted by $F_0^{s/a}$. Note that trivial Leibniz bimodules are symmetric and anti-symmetric. As an immediate consequence of Theorem 4.6 we obtain that the truncated tensor product of a trivial bimodule with an arbitrary bimodule coincides with the "natural" tensor product defined in Section 3:

Corollary 4.8. *Let $\mathfrak{L}$ be a Leibniz algebra over a field $\mathbb{F}$, and let M and N be $\mathfrak{L}$-bimodules. If one of the $\mathfrak{L}$-bimodules is trivial, then $M\overline{\otimes}N = M\underline{\otimes}N = M \otimes N$. In particular,*

$$M\overline{\otimes}F_0^{s/a} = M\underline{\otimes}F_0^{s/a} = M \otimes F_0^{s/a} \cong M \cong F_0^{s/a} \otimes M = F_0^{s/a}\overline{\otimes}M = F_0^{s/a}\underline{\otimes}M.$$

It is not necessarily true that the truncated tensor products are non-zero:

Corollary 4.9. *Let $\mathfrak{L}$ be a Leibniz algebra, and let M and N be non-trivial irreducible $\mathfrak{L}$-bimodules. Then the following statements hold:*

(a) If M is symmetric and N is anti-symmetric, then $M\overline{\otimes}N = M\underline{\otimes}N = 0$.

(b) If M is anti-symmetric and N is symmetric, then $M\overline{\otimes}N = M\underline{\otimes}N = 0$.

Proof. Observe that under either of the two hypotheses we have that $\mathfrak{L}M$ is a non-zero $\mathfrak{L}$-subbimodule of M and $\mathfrak{L}N$ is a non-zero $\mathfrak{L}$-subbimodule of N, and thus we conclude from the irreducibility of either bimodule that $\mathfrak{L}M = M$ and $\mathfrak{L}N = N$. Then (a) is an immediate consequence of Corollary 4.7 (c), and (b) is an immediate consequence of Corollary 4.7 (d). ☐

As promised earlier, we show now that, in general, neither of the truncated tensor products is associative. More precisely, we have the following result:

Proposition 4.10. *For every non-perfect Leibniz algebra $\mathfrak{L}$ there exist $\mathfrak{L}$-bimodules L, M, and N such that*

$$(L\overline{\otimes}M)\overline{\otimes}N \ncong L\overline{\otimes}(M\overline{\otimes}N) \quad \textit{and} \quad (L\underline{\otimes}M)\underline{\otimes}N \ncong L\underline{\otimes}(M\underline{\otimes}N)$$

as $\mathfrak{L}$-bimodules.

Proof. Since by hypothesis $\mathfrak{L}$ is not perfect, there exists a non-zero linear form $\lambda \in \mathfrak{L}^*$ such that $\lambda(\mathfrak{L}\mathfrak{L}) = 0$ (see the argument in Remark 3.11). Hence, the left $\mathfrak{L}$-module $F_{\pm\lambda}$ with underlying vector space $\mathbb{F}$ and action $x \cdot 1 := \pm\lambda(x)$ is non-trivial. Now choose the $\mathfrak{L}$-bimodules $L := F_\lambda^s$, $M := F_{-\lambda}^s$, and $N := F_\lambda^a$. Then we conclude from Theorem 4.6 and Corollary 4.8 that

$$(L\overline{\otimes}M)\overline{\otimes}N \cong (L\underline{\otimes}M)\underline{\otimes}N \cong F_0^{s/a} \otimes N \cong N\,.$$

On the other hand, we obtain from Theorem 4.6 and Corollary 4.9 (a) that

$$L\overline{\otimes}(M\overline{\otimes}N) \cong L\underline{\otimes}(M\underline{\otimes}N) \cong L\underline{\otimes}0 = 0\,,$$

which completes the proof. ☐

In the last section of this paper we will establish the conclusion of Proposition 4.10 for every non-zero Leibniz algebra whose canonical Lie algebra is finite dimensional (see Corollary 5.11).

We conclude this section by discussing some examples of the case where $M = N = \mathfrak{L}_{\mathrm{ad}}$, where $\mathfrak{L}_{\mathrm{ad}}$ denotes the *adjoint* $\mathfrak{L}$*-bimodule* (see [8, Example 3.8]). Note that, in general, the adjoint Leibniz bimodule is neither symmetric nor anti-symmetric. In this special case, we have that

$$\begin{aligned} T(\mathfrak{L}_{\mathrm{ad}}, \mathfrak{L}_{\mathrm{ad}}) \subseteq T_0(\mathfrak{L}_{\mathrm{ad}}, \mathfrak{L}_{\mathrm{ad}}) &:= (\mathfrak{L}_{\mathrm{ad}})_0 \otimes \mathfrak{L}_{\mathrm{ad}}\mathfrak{L} + \mathfrak{L}_{\mathrm{ad}}\mathfrak{L} \otimes (\mathfrak{L}_{\mathrm{ad}})_0 \\ &\subseteq \mathrm{Leib}(\mathfrak{L}) \otimes \mathfrak{L}\mathfrak{L} + \mathfrak{L}\mathfrak{L} \otimes \mathrm{Leib}(\mathfrak{L}) \,, \end{aligned}$$

and the second inclusion is an equality in case the characteristic of the ground field is not 2 (see [8, Example 3.11]). In all the following examples the right-most term in the above chain of inclusions is properly contained in $\mathfrak{L}_{\mathrm{ad}} \otimes \mathfrak{L}_{\mathrm{ad}}$, and therefore the corresponding truncated tensor products are non-zero.

Example 4.11. Let $\mathfrak{A} = \mathbb{F}h \oplus \mathbb{F}e$ be the solvable left Leibniz algebra over a field $\mathbb{F}$ with multiplication $he = e$ and $hh = eh = ee = 0$ (see [8, Example 2.3]). Then we have that $(\mathfrak{A}_{\mathrm{ad}})_0 = \mathrm{Leib}(\mathfrak{A}) = \mathbb{F}e = \mathfrak{A}\mathfrak{A}$, and therefore

$$T(\mathfrak{A}_{\mathrm{ad}}, \mathfrak{A}_{\mathrm{ad}}) = T_0(\mathfrak{A}_{\mathrm{ad}}, \mathfrak{A}_{\mathrm{ad}}) = \mathrm{Leib}(\mathfrak{A}) \otimes \mathfrak{A}\mathfrak{A} + \mathfrak{A}\mathfrak{A} \otimes \mathrm{Leib}(\mathfrak{A}) = \mathbb{F}(e \otimes e) \subsetneqq \mathfrak{A}_{\mathrm{ad}} \otimes \mathfrak{A}_{\mathrm{ad}} \,,$$

as the vector space on the left-hand side is one-dimensional and the vector space on the right-hand side is four-dimensional.

Example 4.12. Let $\mathfrak{N} = \mathbb{F}e \oplus \mathbb{F}c$ be the nilpotent left and right Leibniz algebra over a field $\mathbb{F}$ with multiplication $ee = c$ and $ec = ce = cc = 0$ (see [8, Example 2.4]). Then we have that $\mathrm{Leib}(\mathfrak{N}) = \mathbb{F}c = \mathfrak{N}\mathfrak{N}$, and therefore

$$\mathrm{Leib}(\mathfrak{N}) \otimes \mathfrak{N}\mathfrak{N} + \mathfrak{N}\mathfrak{N} \otimes \mathrm{Leib}(\mathfrak{N}) = \mathbb{F}(c \otimes c) \subsetneqq \mathfrak{N}_{\mathrm{ad}} \otimes \mathfrak{N}_{\mathrm{ad}} \,,$$

as the dimension of the vector space on the left-hand side is 1 and the dimension of the vector space on the right-hand side is 4.

Because of

$$(\mathfrak{N}_{\mathrm{ad}})_0 = \begin{cases} \mathbb{F}c & \text{if } \mathrm{char}(\mathbb{F}) \neq 2 \\ 0 & \text{if } \mathrm{char}(\mathbb{F}) = 2 \,, \end{cases}$$

we obtain that

$$T(\mathfrak{N}_{\mathrm{ad}}, \mathfrak{N}_{\mathrm{ad}}) = \begin{cases} \mathbb{F}(c \otimes c) & \text{if } \mathrm{char}(\mathbb{F}) \neq 2 \\ 0 & \text{if } \mathrm{char}(\mathbb{F}) = 2 \end{cases}$$

and

$$T_0(\mathfrak{N}_{\mathrm{ad}}, \mathfrak{N}_{\mathrm{ad}}) = (\mathfrak{N}_{\mathrm{ad}})_0 \otimes \mathfrak{N}_{\mathrm{ad}}\mathfrak{N} + \mathfrak{N}_{\mathrm{ad}}\mathfrak{N} \otimes (\mathfrak{N}_{\mathrm{ad}})_0 = \begin{cases} \mathbb{F}(c \otimes c) & \text{if } \mathrm{char}(\mathbb{F}) \neq 2 \\ 0 & \text{if } \mathrm{char}(\mathbb{F}) = 2 \,. \end{cases}$$

Hence, we still have that

$$T(\mathfrak{N}_{\mathrm{ad}},\mathfrak{N}_{\mathrm{ad}})=T_0(\mathfrak{N}_{\mathrm{ad}},\mathfrak{N}_{\mathrm{ad}})$$

in any characteristic.

Example 4.13. Consider the hemi-semidirect product $\mathfrak{S}=\mathfrak{sl}_2(\mathbb{C})\ltimes_{\mathrm{hemi}}L(1)$, where $\mathfrak{sl}_2(\mathbb{C})$ is the three-dimensional simple Lie algebra of traceless 2×2 matrices with complex coefficients and $L(1)$ is the two-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module (see [8, Example 2.5]). Then $\mathfrak{S}$ is a simple Leibniz algebra (see [9, Theorem 2.3]), and we have that $\mathrm{Leib}(\mathfrak{S})=L(1)$ and $\mathfrak{S}\mathfrak{S}=\mathfrak{S}$ (for the former see the proof of [9, Theorem 2.3] and for the latter see [8, Proposition 7.1]), and therefore

$$T_0(\mathfrak{S}_{\mathrm{ad}},\mathfrak{S}_{\mathrm{ad}})=\mathrm{Leib}(\mathfrak{S})\otimes\mathfrak{S}\mathfrak{S}+\mathfrak{S}\mathfrak{S}\otimes\mathrm{Leib}(\mathfrak{S})=L(1)\otimes\mathfrak{S}+\mathfrak{S}\otimes L(1)\subsetneqq\mathfrak{S}_{\mathrm{ad}}\otimes\mathfrak{S}_{\mathrm{ad}}\,,$$

as the dimension of the vector space on the left-hand side is

$$\begin{aligned}\dim_{\mathbb{C}}[L(1)\otimes\mathfrak{S}+\mathfrak{S}\otimes L(1)] &= 2\dim_{\mathbb{C}}[L(1)\otimes\mathfrak{S}]-\dim_{\mathbb{C}}[L(1)\otimes\mathfrak{S}\cap\mathfrak{S}\otimes L(1)]\\ &\le 2\dim_{\mathbb{C}}[L(1)\otimes\mathfrak{S}]=20\end{aligned}$$

and the dimension of the vector space on the right-hand side is

$$\dim_{\mathbb{C}}(\mathfrak{S}\otimes\mathfrak{S})=25\,.$$

By a straightforward but tedious computation one can show that, in this case, the equality $T(\mathfrak{S}_{\mathrm{ad}},\mathfrak{S}_{\mathrm{ad}})=T_0(\mathfrak{S}_{\mathrm{ad}},\mathfrak{S}_{\mathrm{ad}})$ also holds.

It would be very interesting to find general sufficient conditions on M and N which guarantee that $M\overline{\otimes}N\neq 0$ or $M\underline{\otimes}N\neq 0$. (Note that $M\underline{\otimes}N\neq 0$ implies $M\overline{\otimes}N\neq 0$.)

5 Grothendieck rings

Let $\mathcal{C}$ be an $\mathbb{F}$-linear abelian category such that every object has a composition series of finite length, and let $\mathrm{Gr}(\mathcal{C})$ denote the free abelian group generated by the set of isomorphism classes $\mathrm{Irr}(\mathcal{C})$ of irreducible (= simple) objects in $\mathcal{C}$. Then we assign to every object X in $\mathcal{C}$ its class[6] $[X]\in\mathrm{Gr}(\mathcal{C})$ by

$$[X]:=\sum_{S\in\mathrm{Irr}(\mathcal{C})}[X:S][S]\,,$$

where $[X:S]$ denotes the (unique) multiplicity of the irreducible object S in a composition series of X and $[S]\in\mathrm{Gr}(\mathcal{C})$ denotes the isomorphism class of S (see [5, Definition 1.5.8]).

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[6] Note that two objects in $\mathcal{C}$ belong to the same class exactly when they have the same composition factors (up to isomorphism) with the same multiplicities. In particular, isomorphic objects in $\mathcal{C}$ belong to the same class, but, of course, unless the objects are irreducible, the converse is not true.

By virtue of the Jordan-Hölder theorem, the above sum is always finite, and $[Z] = [X]+[Y]$ for all short exact sequences $0 \to X \to Z \to Y \to 0$ in $\mathcal{C}$. In particular, we have that

$$[X \oplus Y] = [X] + [Y]\,,$$

i.e., the addition in $\mathrm{Gr}(\mathcal{C})$ corresponds to taking direct sums in $\mathcal{C}$.

Now assume that $\mathcal{C}$ is a multiring category over a field $\mathbb{F}$ (see [5, Definition 4.2.3]). Then the tensor product on $\mathcal{C}$ induces a multiplication on $\mathrm{Gr}(\mathcal{C})$ via

$$[S_1] \cdot [S_2] := [S_1 \otimes S_2] := \sum_{[S] \in \mathrm{Irr}(\mathcal{C})} [S_1 \otimes S_2 : S][S]$$

for all classes $[S_1], [S_2] \in \mathrm{Irr}(\mathcal{C})$. It follows from [5, Lemma 4.5.1] that the multiplication on $\mathrm{Gr}(\mathcal{C})$ is associative, and then $\mathrm{Gr}(\mathcal{C})$ is called the *Grothendieck ring* of $\mathcal{C}$ (see [5, Definition 4.5.2]).

Next, we specialize the previous paragraph to the category $\mathcal{C} := \ \mathrm{mod}\,(\mathfrak{g})$ of finite-dimensional $\mathfrak{g}$-modules for a Lie algebra $\mathfrak{g}$ and to the category $\mathcal{C} := \mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ of finite-dimensional weak $\mathfrak{L}$-bimodules for a Leibniz algebra $\mathfrak{L}$. For brevity we set

$$\mathrm{Gr}(\mathfrak{g}) := \mathrm{Gr}(\mathbf{mod}(\mathfrak{g}))$$

and

$$\mathrm{Gr}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L}) := \mathrm{Gr}(\mathbf{mod}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L}))\,.$$

Although the category $\mathbf{mod}^{\mathrm{bi}}(\mathfrak{L})$ of finite-dimensional $\mathfrak{L}$-bimodules is not a monoidal category, it is still a locally finite $\mathbb{F}$-linear abelian category, and therefore we can define the Grothendieck group $\mathrm{Gr}(\mathbf{mod}^{\mathrm{bi}}(\mathfrak{L}))$ in the same way as above which we will denote by $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$. Moreover, the truncated tensor product $\overline{\otimes}$ from Section 4 induces a multiplication on $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ via

$$[M] \cdot [N] := [M \overline{\otimes} N] := \sum_{[L] \in \mathrm{Irr}^{\mathrm{bi}}(\mathfrak{L})} [M \overline{\otimes} N : L][L]$$

for all classes $[M], [N] \in \mathrm{Irr}^{\mathrm{bi}}(\mathfrak{L})$, where $\mathrm{Irr}^{\mathrm{bi}}(\mathfrak{L})$ denotes the set of isomorphism classes of finite-dimensional irreducible $\mathfrak{L}$-bimodules. Since irreducible Leibniz bimodules are either symmetric or anti-symmetric, it follows from Theorem 4.6 that $M \overline{\otimes} N = M \underline{\otimes} N$ for irreducible $\mathfrak{L}$-bimodules M and N, and therefore instead of $\overline{\otimes}$ we could also have used $\underline{\otimes}$ in the definition of the multiplication of $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$, i.e., for the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ it does not matter which of the truncated tensor products we use.

As neither of the truncated tensor products is associative (see Proposition 4.10), we cannot expect the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ to be associative, and indeed, we will see later in this section that this is quite often not the case (see Proposition 5.10). In fact, from the latter we will deduce Corollary 5.11 which complements Proposition 4.10.

On the other hand, it is an immediate consequence of Proposition 4.4 and Corollary 4.8 that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is commutative and unital:

Proposition 5.1. *The Grothendieck ring* $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ *of a Leibniz algebra* $\mathfrak{L}$ *is commutative. Moreover, the class* $[F_0^{s/a}]$ *of the one-dimensional trivial* $\mathfrak{L}$*-bimodule* $F_0^{s/a}$ *is the unity of* $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$.

We continue by describing the algebraic structure of the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ of a Leibniz algebra $\mathfrak{L}$ as a certain amalgamated product of two copies of the Grothendieck ring $\mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}})$ of the canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ associated with $\mathfrak{L}$, which we call the unital commutative product.

Let A and B be unital commutative (but not necessarily associative) rings whose underlying additive abelian groups are free, and let 1_A resp. 1_B denote the unity of the ring A resp. B. Furthermore, suppose that A_1 is a basis of the free abelian additive group of A such that $1_A \in A_1$ and B_1 is a basis of the free abelian additive group of B such that $1_B \in B_1$. Then we define the *unital commutative product* $A \circledast B$ of A and B as follows:

Set $\widetilde{A} := A_1 \setminus \{1_A\}$ and $\widetilde{B} := B_1 \setminus \{1_B\}$. Adjoin to the set $\widetilde{A} \cup \widetilde{B}$ a new unity 1 and consider the free abelian group $G(A, B)$ generated by this set. Then define a multiplication on the basis of $G(A, B)$ by

$$a * a' := aa' \text{ for all } a, a' \in \widetilde{A} \tag{8}$$

$$b * b' := bb' \text{ for all } b, b' \in \widetilde{B} \tag{9}$$

$$a * b := b * a := 0 \text{ for all } a \in \widetilde{A}\, b \in \widetilde{B} \tag{10}$$

$$1 * x := x * 1 := x \text{ for every } x \in \widetilde{A} \cup \widetilde{B} \cup \{1\} \tag{11}$$

and extend it biadditively to all of $G(A, B)$. While doing so, it is understood that in case $aa' = n1_A + \sum_{\tilde{a} \in \widetilde{A}} n_{\tilde{a}} \tilde{a}$ in A for some non-zero integer n, we require $a * a' = n1 + \sum_{\tilde{a} \in \widetilde{A}} n_{\tilde{a}} \tilde{a}$ in $A \circledast B$, and similarly for B[7].

Observe that $\widetilde{A} \cup \{1\}$ generates a subring of $A \circledast B$ isomorphic to A and that $\widetilde{B} \cup \{1\}$ generates a subring of $A \circledast B$ isomorphic to B.

It is clear from the definition of the multiplication that $A \circledast B$ is a unital commutative ring with unity 1:

Proposition 5.2. *If A and B are unital commutative rings, then the unital commutative product $A \circledast B$ is a unital commutative ring with unity* 1.

Note that the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(0)$ of the zero algebra is the ring of integers which is also the smallest example of a unital commutative product $\mathbb{Z} \circledast \mathbb{Z}$. In particular, this unital commutative product is associative. In the following we will see that this is an exception to the rule.

In fact, it turns out that quite often unital commutative products do not satisfy any of the usual associativity properties (associative, alternative, Jordan identity, power-associative). Note that every commutative ring clearly is reflexive and 3rd power-associative. Moreover, a commutative ring is left alternative exactly when it is right alternative.

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[7] It is understood that at most finitely many of the integers $n_{\tilde{a}}$ are non-zero.

Consequently, a commutative ring that is not alternative is neither left nor right alternative[8]. For later applications, we need the following criterion, which gives sufficient conditions for the failure of each of the first three of the associativity properties mentioned above.

Lemma 5.3. *Let A and B be unital commutative rings. Then the following statements hold:*

(a) If there exist elements $a, a' \in \widetilde{A}$ such that $aa' = n1_A + \sum_{\tilde{a}\in\widetilde{A}} n_{\tilde{a}}\tilde{a}$ for some non-zero integer n and $\widetilde{B} \neq \emptyset$, then $A \circledast B$ is not associative.

(b) If there exists an element $a \in \widetilde{A}$ such that $aa = n1_A + \sum_{\tilde{a}\in\widetilde{A}} n_{\tilde{a}}\tilde{a}$ for some non-zero integer n and $\widetilde{B} \neq \emptyset$, then $A \circledast B$ is not alternative.

(c) If there exist elements $a \in \widetilde{A}$ and $b \in \widetilde{B}$ such that $aa = m1_A + \sum_{\tilde{a}\in\widetilde{A}} m_{\tilde{a}}\tilde{a}$ and at least one of the integers $m_{\tilde{a}} \neq 0$ is non-zero as well as $bb = n1_B + \sum_{\tilde{b}\in\widetilde{B}} n_{\tilde{b}}\tilde{b}$ for some non-zero integer n, then $A \circledast B$ is not a Jordan ring.

Proof. (a): Since by hypothesis $\widetilde{B} \neq \emptyset$, there exists an element $b \in \widetilde{B}$. Then one can build the following expressions:

$$(a * a') * b = \left(n1 + \sum_{\tilde{a}\in\widetilde{A}} n_{\tilde{a}}\tilde{a} \right) * b = nb \neq 0$$

and

$$a * (a' * b) = a * 0 = 0\,,$$

which contradicts associativity.

The proof of (b) is very similar to the proof of (a) and is left to the reader.

(c): If we set $a^2 := aa$, we obtain, similar to the proof of (a), that

$$(a^2 * b) * b = (mb) * b = mbb = (mn)1_B + \sum_{\tilde{b}\in\widetilde{B}} (mn_{\tilde{b}})\tilde{b}\,.$$

On the other hand, we have that

$$\begin{aligned} a^2 * (b * b) &= a^2 * (bb) = \left(m1_A + \sum_{\tilde{a}\in\widetilde{A}} m_{\tilde{a}}\tilde{a} \right) * \left(n1_B + \sum_{\tilde{b}\in\widetilde{B}} n_{\tilde{b}}\tilde{b} \right) \\ &= (mn)1 + \sum_{\tilde{b}\in\widetilde{B}} (mn_{\tilde{b}})\tilde{b} + \sum_{\tilde{a}\in\widetilde{A}} (nm_{\tilde{a}})\tilde{a}\,. \end{aligned}$$

Suppose now that $A \circledast B$ is a Jordan ring. Then $(a^2 * b) * b = a^2 * (b * b)$ implies that $nm_{\tilde{a}} = 0$ for every $\tilde{a} \in \widetilde{A}$, which contradicts the hypothesis. ☐

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[8] See [4] or [21] for the precise definitions of the associativity properties.

Recall that every irreducible $\mathfrak{L}$-bimodule is either the symmetrization M^s or the anti-symmetrization M^a of some irreducible $\mathfrak{L}_{\mathrm{Lie}}$-module M. Hence,

$$\{[M^s] \mid [M] \in \mathrm{Irr}(\mathfrak{L}_{\mathrm{Lie}})\} \cup \{[M^a] \mid [M] \in \mathrm{Irr}(\mathfrak{L}_{\mathrm{Lie}})\}$$

is a basis of the additive abelian group underlying the ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$, where $\mathrm{Irr}(\mathfrak{L}_{\mathrm{Lie}})$ denotes the set of isomorphism classes of finite-dimensional irreducible $\mathfrak{L}_{\mathrm{Lie}}$-modules.

Theorem 5.4. *The Grothendieck ring* $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ *of a Leibniz algebra* $\mathfrak{L}$ *is isomorphic to the unital commutative product of two copies of the Grothendieck ring* $\mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}})$ *of the canonical Lie algebra* $\mathfrak{L}_{\mathrm{Lie}}$ *associated with* $\mathfrak{L}$*, i.e.,*

$$\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L}) \cong \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}}) \circledast \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}})$$

as unital commutative rings.

Proof. Let A denote the free abelian subgroup of $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ that generated by

$$\{[M^s] \mid [M] \in \mathrm{Irr}(\mathfrak{L}_{\mathrm{Lie}})\},$$

and let B denote the free abelian subgroup of $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ that is generated by

$$\{[M^a] \mid [M] \in \mathrm{Irr}(\mathfrak{L}_{\mathrm{Lie}})\}.$$

It follows from Corollary 4.7 (a) and (b) that

$$M^s \overline{\otimes} N^s = M^s \otimes N^s = (M \otimes N)^s$$

and

$$M^a \overline{\otimes} N^a = M^a \otimes N^a = (M \otimes N)^a$$

for all irreducible $\mathfrak{L}_{\mathrm{Lie}}$-modules M and N. This shows that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ contains A and B as unital subrings. In particular, the relations (8) and (9) hold in $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$. Moreover, we have that $A \cong \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}})$ via the isomorphism $[M] \mapsto [M^s]$ and $B \cong \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}})$ via the isomorphism $[M] \mapsto [M^a]$.

In addition, we obtain from Corollary 4.9 that

$$M^s \overline{\otimes} N^a = M^a \overline{\otimes} N^s = 0$$

for all non-trivial irreducible $\mathfrak{L}_{\mathrm{Lie}}$-modules M and N. This implies that (10) holds in $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$.

Finally, (11) is an immediate consequence of Corollary 4.8. □

In particular, Theorem 5.4 shows that the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ only depends on the canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$. By applying Theorem 5.4 both to $\mathfrak{L}$ and $\mathfrak{L}_{\mathrm{Lie}}$, we obtain the following result, which shows that for computing the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ it is enough to consider only Lie algebras:

Corollary 5.5. *The Grothendieck ring* $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ *of a Leibniz algebra* $\mathfrak{L}$ *is isomorphic to the Grothendieck ring* $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L}_{\mathrm{Lie}})$ *of the canonical Lie algebra* $\mathfrak{L}_{\mathrm{Lie}}$ *associated with* $\mathfrak{L}$*, i.e.,*

$$\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L}) \cong \mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L}_{\mathrm{Lie}})$$

as unital commutative rings.

Next, we illustrate Theorem 5.4 and Corollary 5.5 by the following example.

Example 5.6. Let $\mathfrak{e} := \mathbb{F}e$ be the one-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of arbitrary characteristic. Then the finite-dimensional irreducible $\mathfrak{e}$-modules are the one-dimensional modules F_λ, where e acts as multiplication by a scalar λ on the ground field $\mathbb{F}$, and therefore the isomorphism classes of the finite-dimensional irreducible $\mathfrak{e}$-bimodules are

$$\{[F_\lambda^s] \mid \lambda \in \mathbb{F}\} \cup \{[F_\mu^a] \mid \mu \in \mathbb{F}\}\,.$$

By virtue of Corollary 4.7 (a), (b) and Corollary 4.9, the truncated tensor products are either the "natural" tensor product defined in Section 3 or zero, i.e.,

$$F_\lambda^s \overline{\otimes} F_\mu^s \cong F_{\lambda+\mu}^s\,, \qquad F_\lambda^a \overline{\otimes} F_\mu^a \cong F_{\lambda+\mu}^a\,, \qquad F_\lambda^s \overline{\otimes} F_\mu^a \cong F_\mu^a \overline{\otimes} F_\lambda^s = 0\,,$$

where in the last isomorphism we suppose that $\lambda \neq 0$ and $\mu \neq 0$. Moreover, it follows from Corollary 4.8 that the one-dimensional trivial $\mathfrak{e}$-bimodule $F_0^{s/a} := F_0^s = F_0^a$ acts as an identity for both truncated tensor products. As a consequence, we obtain that the Grothendieck ring $\mathrm{Gr}(\mathfrak{e}) = \bigoplus_{\lambda\in\mathbb{F}} \mathbb{Z}[F_\lambda]$ of the Lie algebra $\mathfrak{e}$ is isomorphic to the integral group ring $\mathbb{Z}[\mathbb{F}^+]$ of the additive group $\mathbb{F}^+$ of the ground field $\mathbb{F}$, and thus it follows from Theorem 5.4 that the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{e})$ is the unital commutative product of two copies of $\mathbb{Z}[\mathbb{F}^+]$:

$$\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{e}) \cong \mathbb{Z}[\mathbb{F}^+] \circledast \mathbb{Z}[\mathbb{F}^+]\,,$$

where one copy of $\mathbb{Z}[\mathbb{F}^+]$ corresponds to the symmetrizations of the finite-dimensional irreducible $\mathfrak{e}$-modules and the other copy corresponds to the anti-symmetrizations of the finite-dimensional irreducible $\mathfrak{e}$-modules.

More generally, it then follows from Corollary 5.5 that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L}) \cong \mathbb{Z}[\mathbb{F}^+] \circledast \mathbb{Z}[\mathbb{F}^+]$ for every Leibniz algebra $\mathfrak{L}$ whose canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ is one-dimensional. In particular, this applies to every two-dimensional non-Lie Leibniz algebra.

Note that the classes $\pm[F_\lambda^s]$, $\pm[F_\lambda^a]$ ($\lambda \in \mathbb{F}$) are at the same time multiplicatively invertible and zero divisors. This already shows that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{e})$ is not associative because in a unital commutative associative ring multiplicatively invertible elements are never zero divisors and vice versa. But this can also be seen more explicitly as follows (cf. also the proof of Proposition 4.10). Indeed, by fixing a non-zero scalar $\lambda \in \mathbb{F}$, we have that

$$([F_\lambda^s] \cdot [F_{-\lambda}^s]) \cdot [F_\lambda^a] = [F_0^{s/a}] \cdot [F_\lambda^a] = [F_\lambda^a]\,.$$

On the other hand, we have that

$$[F_\lambda^s] \cdot ([F_{-\lambda}^s] \cdot [F_\lambda^a]) = [F_\lambda^s] \cdot 0 = 0\,,$$

which again shows that the multiplication of $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{e})$ is not associative.

However, we will see in Theorem 5.13 and Corollary 5.17 below that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{e})$ is an alternative power-associative Jordan ring.

By employing Lie's theorem (see [13, Theorem 4.1]), we can generalize Example 5.6 considerably for ground fields of characteristic zero.

Theorem 5.7. *Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero. Then*

$$\mathrm{Gr}(\mathfrak{g}) \cong \mathbb{Z}[\mathbb{F}^+ \times \cdots \times \mathbb{F}^+]$$

as unital commutative rings, where $\mathbb{F}^+$ denotes the additive group of $\mathbb{F}$ and the number of factors $\mathbb{F}^+$ in the group ring is $\dim_{\mathbb{F}} \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$[9].

Proof. According to Lie's theorem, every finite-dimensional irreducible $\mathfrak{g}$-module is one-dimensional. Since the derived subalgebra $[\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$ acts trivially on every one-dimensional $\mathfrak{g}$-module, the finite-dimensional irreducible $\mathfrak{g}$-modules are in bijection with $(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$ from which the assertion follows similarly to the argument in Example 5.6. ☐

As an immediate consequence of Theorem 5.4 and Theorem 5.7 we obtain the following result:

Corollary 5.8. *Let $\mathfrak{L}$ be a solvable Leibniz algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero whose canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ is finite dimensional. Then*

$$\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L}) \cong \mathbb{Z}[\mathbb{F}^+ \times \cdots \times \mathbb{F}^+] \circledast \mathbb{Z}[\mathbb{F}^+ \times \cdots \times \mathbb{F}^+]$$

as unital commutative rings, where $\mathbb{F}^+$ denotes the additive group of $\mathbb{F}$ and the number of factors $\mathbb{F}^+$ in each of the group rings is $\dim_{\mathbb{F}} \mathfrak{L}_{\mathrm{Lie}}/[\mathfrak{L}_{\mathrm{Lie}},\mathfrak{L}_{\mathrm{Lie}}]$.

Next, we prove that the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ of a Leibniz algebra $\mathfrak{L}$ quite often is not associative. For this we need the following result that should be well-known but for which we could not find a reference.

Lemma 5.9. *Every non-zero finite-dimensional Lie algebra has a finite-dimensional non-trivial irreducible module.*

Proof. Let $\mathfrak{g}$ be a non-zero finite-dimensional Lie algebra and suppose that the trivial module is the only finite-dimensional irreducible $\mathfrak{g}$-module. But then it follows from [7, Proposition 1] that $\mathfrak{g}$ is nilpotent. As $\mathfrak{g} \neq 0$, $\mathfrak{g}$ is not perfect, and thus there exists a non-zero linear form $\lambda \in \mathfrak{g}^*$ such that $\lambda([\mathfrak{g},\mathfrak{g}]) = 0$ (see the argument in Remark 3.11). Hence, the $\mathfrak{g}$-module F_λ with underlying vector space $\mathbb{F}$ and $\mathfrak{g}$-action $x \cdot 1 := \lambda(x)$ clearly defines a non-trivial finite-dimensional irreducible $\mathfrak{g}$-module contradicting our assumption. ☐

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[9] Here $[\mathfrak{g},\mathfrak{g}] := \langle [x,y] \in \mathfrak{g} \mid x,y \in \mathfrak{g}\rangle_{\mathbb{F}}$ is the *derived subalgebra* of the Lie algebra $\mathfrak{g}$.

Proposition 5.10. *Let $\mathfrak{L}$ be a non-zero Leibniz algebra whose canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ is finite dimensional. Then the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is not associative.*

Proof. Since $\mathfrak{L} \neq 0$, we obtain from [8, Proposition 2.20] that $\mathfrak{L}_{\mathrm{Lie}} \neq 0$. Then it follows from Lemma 5.9 that there exists a finite-dimensional non-trivial irreducible $\mathfrak{L}_{\mathrm{Lie}}$-module M, and thus M^* is also a non-trivial irreducible $\mathfrak{L}_{\mathrm{Lie}}$-module. Hence, the contraction $M^* \otimes M \to F_0$, $\mu \otimes m \mapsto \mu(m)$ is an epimorphism of $\mathfrak{L}_{\mathrm{Lie}}$-modules, where F_0 denotes the one-dimensional trivial $\mathfrak{L}_{\mathrm{Lie}}$-module. In particular, we have that $[M^* \otimes M : F_0] \neq 0$, and we deduce from the definition of the multiplication in $\mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}})$ that

$$[M^*] \cdot [M] = \sum_{[L] \in \mathrm{Irr}(\mathfrak{L}_{\mathrm{Lie}})} [M^* \otimes M : L][L]\,.$$

Hence, the assertion follows from Theorem 5.4 and Lemma 5.3 (a). □

As an immediate consequence of Proposition 5.10 we obtain the next result which complements Proposition 4.10 (see Example 5.12 below):

Corollary 5.11. *Let $\mathfrak{L}$ be a non-zero Leibniz algebra whose canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ is finite dimensional. Then there exist $\mathfrak{L}$-bimodules L, M, and N such that*

$$(L \overline{\otimes} M) \overline{\otimes} N \ncong L \overline{\otimes} (M \overline{\otimes} N)$$

and

$$(L \underline{\otimes} M) \underline{\otimes} N \ncong L \underline{\otimes} (M \underline{\otimes} N)$$

as $\mathfrak{L}$-bimodules.

Example 5.12. Let $\mathfrak{sl}_2(\mathbb{C})$ be the three-dimensional simple Lie algebra of traceless complex 2×2 matrices, and let $M(\lambda)$ denote the Verma module of highest weight λ which is an infinite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. (Here we identify every complex multiple of the unique fundamental weight with its coefficient.) It is well known (see [13, Exercise 7 (c) in Section 7]) that $M(\lambda)$ is irreducible if $\lambda + 1$ is not a dominant integral weight (i.e., with our identification, $\lambda + 1$ is not a non-negative integer). Hence, it follows from [9, Theorem 2.3] in conjunction with [8, Proposition 7.1] that the hemi-semidirect product $\mathfrak{sl}_2(\mathbb{C}) \ltimes_{\mathrm{hemi}} M(-2)$ is a perfect Leibniz algebra whose canonical Lie algebra is finite dimensional.[10] Consequently, Corollary 5.11 applies, but Proposition 4.10 does not.

On the other hand, for every infinite-dimensional non-perfect Lie algebra (for example, the infinite-dimensional Heisenberg algebra) Proposition 4.10 applies, but Corollary 5.11 does not.

Similar to the proof of Corollary 5.8, we can employ Lie's theorem to show that the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ of a finite-dimensional solvable Leibniz algebra $\mathfrak{L}$ over an algebraically closed field of characteristic zero is alternative, or even slightly more general:

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[10] This example also shows that the hypothesis in Corollary 5.11 (and in several other results in this section) is weaker than just to assume that the Leibniz algebra itself is finite dimensional.

Theorem 5.13. *Let $\mathfrak{L}$ be a solvable Leibniz algebra over an algebraically closed field of characteristic zero whose canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ is finite dimensional. Then the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is alternative.*

Proof. Since by Proposition 5.1, $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is commutative, it suffices to prove that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is left alternative. By virtue of Lie's theorem, every finite-dimensional irreducible $\mathfrak{L}_{\mathrm{Lie}}$-module is one-dimensional. Hence, we have that

$$\mathrm{Irr}(\mathfrak{L}_{\mathrm{Lie}}) = \{[F_\lambda] \mid \lambda \in \Lambda\}\,,$$

where

$$\Lambda := \{\lambda \in \mathfrak{L}_{\mathrm{Lie}}^* \mid \lambda([\mathfrak{L}_{\mathrm{Lie}}, \mathfrak{L}_{\mathrm{Lie}}]) = 0\}\,,$$

and therefore

$$\mathrm{Irr}^{\mathrm{bi}}(\mathfrak{L}) = \{[F_0^{s/a}]\} \cup \{[F_\lambda^s] \mid \lambda \in \Lambda \setminus \{0\}\} \cup \{[F_\mu^a] \mid \lambda \in \Lambda \setminus \{0\}\}\,.$$

Let

$$u = m_0[F_0^{s/a}] + \sum_{\lambda\in\Lambda\setminus\{0\}} m_\lambda[F_\lambda^s] + \sum_{\mu\in\Lambda\setminus\{0\}} m_\mu[F_\mu^a]$$

and

$$v = n_0[F_0^{s/a}] + \sum_{\nu\in\Lambda\setminus\{0\}} n_\nu[F_\nu^s] + \sum_{\eta\in\Lambda\setminus\{0\}} n_\eta[F_\eta^a]$$

be arbitrary elements in $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$. Then we have that

$$\begin{aligned} u^2 &= m_0^2[F_0^{s/a}] + \sum_{\lambda\in\Lambda\setminus\{0\}} (2m_0m_\lambda)[F_\lambda^s] + \sum_{\mu\in\Lambda\setminus\{0\}} (2m_0m_\mu)[F_\mu^a] \\ &\quad + \sum_{\lambda',\lambda''\in\Lambda\setminus\{0\}} (m_{\lambda'}m_{\lambda''})[F_{\lambda'+\lambda''}^s] + \sum_{\mu',\mu''\in\Lambda\setminus\{0\}} (m_{\mu'}m_{\mu''})[F_{\mu'+\mu''}^a]\,, \end{aligned}$$

and thus

$$\begin{aligned} u^2\cdot v &= (m_0^2n_0)[F_0^{s/a}] + \sum_{\nu\in\Lambda\setminus\{0\}} (m_0^2n_\nu)[F_\nu^s] + \sum_{\eta\in\Lambda\setminus\{0\}} (m_0^2n_\eta)[F_\eta^a] \\ &\quad + \sum_{\lambda\in\Lambda\setminus\{0\}} (2m_0m_\lambda n_0)[F_\lambda^s] + \sum_{\lambda,\nu\in\Lambda\setminus\{0\}} (2m_0m_\lambda n_\nu)[F_{\lambda+\nu}^s] \\ &\quad + \sum_{\mu\in\Lambda\setminus\{0\}} (2m_0m_\mu n_0)[F_\mu^a] + \sum_{\mu,\eta\in\Lambda\setminus\{0\}} (2m_0m_\mu n_\eta)[F_{\mu+\eta}^a] \\ &\quad + \sum_{\lambda',\lambda''\in\Lambda\setminus\{0\}} (m_{\lambda'}m_{\lambda''}n_0)[F_{\lambda'+\lambda''}^s] + \sum_{\lambda',\lambda'',\eta\in\Lambda\setminus\{0\}} (m_{\lambda'}m_{\lambda''}n_\nu)[F_{\lambda'+\lambda''+\nu}^s] \\ &\quad + \sum_{\mu',\mu''\in\Lambda\setminus\{0\}} (m_{\mu'}m_{\mu''}n_0)[F_{\mu'+\mu''}^a]) + \sum_{\mu',\mu'',\eta\in\Lambda\setminus\{0\}} (m_{\mu'}m_{\mu''}n_\eta)[F_{\mu'+\mu''+\eta}^a])\,. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} u \cdot v &= (m_0 n_0)[F_0^{s/a}] + \sum_{\nu\in\Lambda\setminus\{0\}} (m_0 n_\nu)[F_\nu^s] + \sum_{\eta\in\Lambda\setminus\{0\}} (m_0 n_\eta)[F_\eta^a] \\ &\quad + \sum_{\lambda\in\Lambda\setminus\{0\}} (m_\lambda n_0)[F_\lambda^s] + \sum_{\lambda,\nu\in\Lambda\setminus\{0\}} (m_\lambda n_\nu)[F_{\lambda+\nu}^s] \\ &\quad + \sum_{\mu\in\Lambda\setminus\{0\}} (m_\mu n_0)[F_\mu^a] + \sum_{\mu,\eta\in\Lambda\setminus\{0\}} (m_\mu n_\eta)[F_{\mu+\eta}^a] , \end{aligned}$$

and then

$$\begin{aligned} u \cdot (u \cdot v) &= (m_0^2 n_0)[F_0^{s/a}] + \sum_{\nu\in\Lambda\setminus\{0\}} (m_0^2 n_\nu)[F_\nu^s] + \sum_{\eta\in\Lambda\setminus\{0\}} (m_0^2 n_\eta)[F_\eta^a] \\ &\quad + \underbrace{\sum_{\lambda\in\Lambda\setminus\{0\}} (m_0 m_\lambda n_0)[F_\lambda^s]}_{1} + \underbrace{\sum_{\lambda,\nu\in\Lambda\setminus\{0\}} (m_0 m_\lambda n_\nu)[F_{\lambda+\nu}^s]}_{2} \\ &\quad + \underbrace{\sum_{\mu\in\Lambda\setminus\{0\}} (m_0 m_\mu n_0)[F_\mu^a]}_{3} + \underbrace{\sum_{\mu,\eta\in\Lambda\setminus\{0\}} (m_0 m_\mu n_\eta)[F_{\mu+\eta}^a]}_{4} \\ &\quad + \underbrace{\sum_{\lambda\in\Lambda\setminus\{0\}} (m_\lambda m_0 n_0)[F_\lambda^s]}_{1} + \underbrace{\sum_{\lambda,\nu\in\Lambda\setminus\{0\}} (m_\lambda m_0 n_\nu)[F_{\lambda+\nu}^s]}_{2} \\ &\quad + \sum_{\lambda',\lambda''\in\Lambda\setminus\{0\}} (m_{\lambda'} m_{\lambda''} n_0)[F_{\lambda'+\lambda''}^s] + \sum_{\lambda',\lambda'',\nu\in\Lambda\setminus\{0\}} (m_{\lambda'} m_{\lambda''} n_\nu)[F_{\lambda'+\lambda''+\nu}^s] \\ &\quad + \underbrace{\sum_{\mu\in\Lambda\setminus\{0\}} (m_\mu m_0 n_0)[F_\mu^a]}_{3} + \underbrace{\sum_{\mu,\eta\in\Lambda\setminus\{0\}} (m_\mu m_0 n_\eta)[F_{\mu+\eta}^a]}_{4} \\ &\quad + \sum_{\mu',\mu''\in\Lambda\setminus\{0\}} (m_{\mu'} m_{\mu''} n_0)[F_{\mu'+\mu''}^a] + \sum_{\mu',\mu'',\eta\in\Lambda\setminus\{0\}} (m_{\mu'} m_{\mu''} n_\eta)[F_{\mu'+\mu''+\eta}^a] \\ &= (m_0^2 n_0)[F_0^{s/a}] + \sum_{\nu\in\Lambda\setminus\{0\}} (m_0^2 n_\nu)[F_\nu^s] + \sum_{\eta\in\Lambda\setminus\{0\}} (m_0^2 n_\eta)[F_\eta^a] \\ &\quad + \sum_{\lambda\in\Lambda\setminus\{0\}} (2m_0 m_\lambda n_0)[F_\lambda^s] + \sum_{\lambda,\nu\in\Lambda\setminus\{0\}} (2m_0 m_\lambda n_\nu)[F_{\lambda+\nu}^s] \\ &\quad + \sum_{\mu\in\Lambda\setminus\{0\}} (2m_0 m_\mu n_0)[F_\mu^a] + \sum_{\mu,\eta\in\Lambda\setminus\{0\}} (2m_0 m_\mu n_\eta)[F_{\mu+\eta}^a] \\ &\quad + \sum_{\lambda',\lambda''\in\Lambda\setminus\{0\}} (m_{\lambda'} m_{\lambda''} n_0)[F_{\lambda'+\lambda''}^s] + \sum_{\lambda',\lambda'',\nu\in\Lambda\setminus\{0\}} (m_{\lambda'} m_{\lambda''} n_\nu)[F_{\lambda'+\lambda''+\nu}^s] \\ &\quad + \sum_{\mu',\mu''\in\Lambda\setminus\{0\}} (m_{\mu'} m_{\mu''} n_0)[F_{\mu'+\mu''}^a] + \sum_{\mu',\mu'',\eta\in\Lambda\setminus\{0\}} (m_{\mu'} m_{\mu''} n_\eta)[F_{\mu'+\mu''+\eta}^a] , \end{aligned}$$

which completes the proof. ☐

The next example shows that Theorem 5.13 is not true in non-zero characteristic:

Example 5.14. Consider the non-abelian solvable restricted Lie algebra $L = \mathbb{F}t \ltimes I$ in Theorem 3.1 of [10] over an algebraically closed field $\mathbb{F}$ of prime characteristic p and choose Z to be properly contained in T. Then it follows from Corollary 1.2 b) in [6] that each of the p-dimensional irreducible restricted modules is self-dual. Hence, we can argue similar to the proof of Proposition 5.10 by applying part (b) instead of part (a) of Lemma 5.3 to show that the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(L)$ is not alternative.

For the convenience of the reader we include the following well-known result:

Lemma 5.15. *Let R be a commutative ring. Then the following statements hold:*

(a) If R is alternative, then R is a Jordan ring.

(b) If R is a Jordan ring of characteristic $\neq 2, 3, 5$, then R is power-associative.

Proof. (a): Since by hypothesis R is alternative and commutative, we obtain from the left alternative law, i.e., $r^2s = r(rs)$ holds for arbitrary elements $r, s \in R$, in conjunction with the commutativity of R that

$$(u^2v)u = [u(uv)]u = u[u(vu)] = u^2(vu)$$

for all elements $u, v \in R$.

(b): According to [1, Lemmas 3 and 4], it is enough to prove that R is 4^{rd} *power-associative*, i.e., R satisfies $r^2r^2 = (r^2r)r$ for every element $r \in R$. But the latter is a special case of the Jordan identity. ☐

Remark 5.16. Note that Lemma 5.15 suffices for our purposes, but the implications are known to be true more generally. Namely, it follows from a theorem of Emil Artin that every alternative ring is power-associative. Moreover, every (not necessarily commutative) Jordan ring is power-associative (see Fact 6 on p. 19 in [4]).

As an immediate consequence of Theorem 5.13 in conjunction with Proposition 5.1 and Lemma 5.15 we obtain the following result:

Corollary 5.17. *Let $\mathfrak{L}$ be a solvable Leibniz algebra over an algebraically closed field of characteristic zero whose canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ is finite dimensional. Then the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is a power-associative commutative Jordan ring.*

Now, we consider the Grothendieck rings $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ of finite-dimensional semi-simple Leibniz algebras $\mathfrak{L}$ and begin with the smallest possible example:

Example 5.18. Let $\mathfrak{sl}_2(\mathbb{C})$ denote the three-dimensional simple Lie algebra of traceless complex 2×2 matrices. Note that the category $\mathbf{mod}^{\mathrm{bi}}(\mathfrak{sl}_2(\mathbb{C}))$ of finite-dimensional

$\mathfrak{sl}_2(\mathbb{C})$-bimodules has already been studied in [18]. Here we consider the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{sl}_2(\mathbb{C}))$ of this category.

It is well known that the isomorphism classes of the finite-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules are given by

$$\{[L(n)] \mid \dim_{\mathbb{C}} L(n) = n+1\,,\ n \in \mathbb{N}_0\}$$

(see [13, Theorem 7.2]), and therefore the isomorphism classes of finite-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-bimodules are

$$\{[L(n)^s] \mid n \in \mathbb{N}_0\} \cup \{[L(n)^a] \mid n \in \mathbb{N}_0\}\,.$$

By virtue of Corollary 4.7 (a), (b) and Corollary 4.9, the truncated tensor products are either the "natural" tensor product defined in Section 3 or zero, i.e.,

$$L(m)^s \overline{\otimes} L(n)^s \cong [L(m) \otimes L(n)]^s, \qquad L(m)^a \overline{\otimes} L(n)^a \cong [L(m) \otimes L(n)]^a\,,$$
$$L(m)^s \overline{\otimes} L(n)^a \cong L(n)^a \overline{\otimes} L(m)^s = 0\,,$$

where in the last isomorphism we assume that $m > 0$ and $n > 0$. Moreover, the tensor products $L(m) \otimes L(n)$ can be obtained from the Clebsch-Gordan formula (see [13, Exercise 7 in Section 22]). Finally, it follows from Corollary 4.8 that the one-dimensional trivial $\mathfrak{sl}_2(\mathbb{C})$-bimodule $L(0)^{s/a} := L(0)^s = L(0)^a$ acts as an identity for both truncated tensor products.

As a consequence of the Clebsch-Gordan formula, we obtain that the Grothendieck ring $\mathrm{Gr}(\mathfrak{sl}_2(\mathbb{C}))$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to the polynomial ring $\mathbb{Z}[t]$ in one variable generated by the class $[L(1)]$ of the two-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module, and therefore it follows from Theorem 5.4 that the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{sl}_2(\mathbb{C}))$ is the unital commutative product of two copies of $\mathbb{Z}[t]$:

$$\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{sl}_2(\mathbb{C})) \cong \mathbb{Z}[t] \circledast \mathbb{Z}[t]\,,$$

where one copy of $\mathbb{Z}[t]$ corresponds to the symmetrizations of the finite-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules and the other copy corresponds to the anti-symmetrizations of the finite-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules.

More generally, it then follows from Corollary 5.5 that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L}) \cong \mathbb{Z}[t] \circledast \mathbb{Z}[t]$ for every Leibniz algebra $\mathfrak{L}$ whose canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. In particular, this applies to the simple[11] non-Lie Leibniz algebras $\mathfrak{sl}_2(\mathbb{C}) \ltimes_{\mathrm{hemi}} L(n)$ for every positive integer n.

Note also that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{sl}_2(\mathbb{C}))$ is neither alternative nor a Jordan ring. Indeed, we obtain from the Clebsch-Gordan formula that

$$L(m) \otimes L(n) \cong L(m+n) \oplus L(m+n-2) \oplus \cdots \oplus L(m-n)$$

---

[11] See Theorem 2.3 in [9].

for all integers $m \geq n \geq 0$, and where exactly $n+1$ summands occur on the right-hand side. In particular, $L(1) \otimes L(1) \cong L(0) \oplus L(2)$ contains the one-dimensional trivial module and a non-trivial irreducible module as direct summands. In view of Lemma 5.3 (c), this implies in turn that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{sl}_2(\mathbb{C}))$ is not a Jordan ring, and therefore it follows from Lemma 5.15 (a) that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{sl}_2(\mathbb{C}))$ is also not alternative.

Note that, for dimension reasons, every finite-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module is self-dual. In fact, every finite-dimensional non-zero semi-simple Lie algebra over a field of characteristic zero has a finite-dimensional non-trivial self-dual irreducible module, namely, the adjoint module of one of its minimal ideals, and so we can apply part (b) of Lemma 5.3. It turns out that even more is true which enables us to apply part (c) of this lemma. We will use this observation to show that the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ of every finite-dimensional non-zero semi-simple Leibniz algebra over a field of characteristic zero is neither alternative nor a Jordan ring which is contrary to the behavior of solvable Leibniz algebras as we have seen in Theorem 5.13 and Corollary 5.17.

Theorem 5.19. *Let $\mathfrak{L}$ be a non-zero semi-simple Leibniz algebra over a field $\mathbb{F}$ of characteristic zero whose canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ is finite dimensional. Then the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is neither alternative nor a Jordan ring.*

Proof. As in the proof of Proposition 5.10, we have that $\mathfrak{g} := \mathfrak{L}_{\mathrm{Lie}} \neq 0$. Note that it follows from [8, Proposition 7.8] that $\mathfrak{g}$ is necessarily semi-simple. Now let M be a minimal ideal of $\mathfrak{g}$. Then M is a simple Lie algebra (see [13, Theorem 5.2]) and a finite-dimensional non-trivial irreducible $\mathfrak{g}$-module. Since the Killing form of M is $\mathfrak{g}$-invariant (see [13, Lemma 5.1]) and non-degenerate (see [13, Theorem 5.1]), M and its linear dual M^* are isomorphic $\mathfrak{g}$-modules (cf. Proposition 4.6 in Chapter 3 of [22]).

Furthermore, it follows from Weyl's theorem on complete reducibility (see [13, Theorem 6.3]) and the bijection between finite-dimensional irreducible $\mathfrak{g}$-modules and the set of dominant integral weights Λ^+ with respect to a Cartan subalgebra of $\mathfrak{g}$ (see [13, Corollary 21.2]) that

$$M \otimes M = \bigoplus_{\lambda \in \Lambda^+} m_\lambda L(\lambda)\,,$$

where $L(\lambda)$ denotes the finite-dimensional irreducible $\mathfrak{g}$-module of highest weight λ and m_λ is the multiplicity of $L(\lambda)$ in $M \otimes M$. Since M is irreducible, we conclude from Schur's lemma that $\mathrm{End}_{\mathfrak{g}}(M)$ is a division algebra. On the other hand, because $\mathfrak{g}$ is perfect (see [13, Corollary 5.2]) and M is non-trivial, we have that $d := \dim_{\mathbb{F}} M > 1$, and therefore $\mathrm{End}_{\mathbb{F}}(M) \cong M_d(\mathbb{F})$ has zero divisors. Consequently, we obtain that $\mathrm{End}_{\mathfrak{g}}(M) \subsetneqq \mathrm{End}_{\mathbb{F}}(M)$.

Next, we compute m_0 by taking $\mathfrak{g}$-invariants on both sides, and as M is self-dual, we then deduce the following inequality:

$$\begin{aligned} m_0 &= \dim_{\mathbb{F}}(M \otimes M)^{\mathfrak{g}} = \dim_{\mathbb{F}}(M^* \otimes M)^{\mathfrak{g}} \\ &= \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(M, M)^{\mathfrak{g}} = \dim_{\mathbb{F}} \mathrm{End}_{\mathfrak{g}}(M) \\ &< \dim_{\mathbb{F}} \mathrm{End}_{\mathbb{F}}(M) = \dim_{\mathbb{F}}(M \otimes M)\,. \end{aligned}$$

Hence, for dimension reasons, $M \otimes M$ must have a non-trivial irreducible direct summand, and therefore we conclude from Lemma 5.3 (c) that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is not a Jordan ring. Finally, it follows from Lemma 5.15 (a) that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is also not alternative. ☐

Remark 5.20. Unfortunately, we do not know whether the Grothendieck ring in Theorem 5.19 is power-associative or not. But a straightforward computation using the Clebsch-Gordan formula shows that $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{sl}_2(\mathbb{C}))$ is not power-associative, so we suspect this might be true more generally for every finite-dimensional non-zero semi-simple Leibniz algebra in characteristic zero. On the other hand, except for Proposition 5.10, Theorem 5.13, and Corollary 5.17, we do not know anything about the associativity properties of $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ for non-semi-simple Leibniz algebras $\mathfrak{L}$ in characteristic zero. Moreover, we know even less about these associativity properties in non-zero characteristics, but see Proposition 5.10, Example 5.14, and Proposition 5.21 below.

Be aware that, contrary to the situation in characteristic zero, the Killing form (or more generally, trace forms) of simple Lie algebras over a field of non-zero characteristic can be zero (see Theorem 1.3 in Chapter 4 of [22]). But nevertheless, similar to Theorem 5.19, one can deduce the following result which is valid for ground fields of arbitrary characteristic:

Proposition 5.21. *Let $\mathfrak{L}$ be a non-zero Leibniz algebra whose canonical Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$ is finite dimensional simple and admits a non-degenerate invariant bilinear form. Then the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is neither alternative nor a Jordan ring.*

Remark 5.22. Note that the hypothesis of Proposition 5.21 is satisfied in many instances. Namely, for Lie algebras of classical type (see [3, Corollary 6.1] and [12, Theorems A and B]), for graded Lie algebras of Cartan type (see Section 6 in Chapter 4 of [22]), for (non-graded) Block algebras (see [2, Theorem 7]), and for Melikian algebras (see [20, Proposition 6.1]).

We conclude our paper by briefly discussing a possible weak analogue of Theorem 5.4 for the Grothendieck ring $\mathrm{Gr}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ of a Leibniz algebra $\mathfrak{L}$.

Recall that the *pushout* of arbitrary commutative associative rings R_1 and R_2 with unities 1_{R_1} and 1_{R_2}, respectively, is the tensor product $R_1 \otimes_{\mathbb{Z}} R_2$ with multiplication

$$(r_1 \otimes r_2)(r_1' \otimes r_2') := (r_1 r_1') \otimes (r_2 r_2')$$

and unity $(1_{R_1}, 1_{R_2})$.

Let $\mathfrak{L}$ be a Leibniz algebra over a field $\mathbb{F}$. Then we consider the following ring homomorphisms:

$$\iota : \mathbb{Z} \to \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}}),\ 1 \mapsto [F_0],$$
$$\varphi^s : \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}}) \to \mathrm{Gr}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L}),\ [M] \mapsto [M^s],$$
$$\varphi^a : \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}}) \to \mathrm{Gr}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L}),\ [M] \mapsto [M^a].$$

Clearly, we have that $\varphi^s \circ \iota = \varphi^a \circ \iota$, and thus it follows from the universal property of the pushout that there is a unique ring homomorphism

$$\pi : \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}}) \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}}) \to \mathrm{Gr}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$$

such that $\varphi^s = \pi \circ \kappa^s$ and $\varphi^a = \pi \circ \kappa^s$, where $\kappa^s : \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}}) \to \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}}) \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}})$ and $\kappa^a : \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}}) \to \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}}) \otimes_{\mathbb{Z}} \mathrm{Gr}(\mathfrak{L}_{\mathrm{Lie}})$ are the defining ring homomorphisms of the pushout. Note that one cannot even hope that π is an epimorphism as it is not even an epimorphism of abelian groups (see our discussion of a possible classification of finite-dimensional irreducible weak Leibniz bimodules in Section 3).

On the other hand, $\Phi : \mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L}) \to \mathrm{Gr}^{\mathrm{bi}}_{\mathrm{weak}}(\mathfrak{L})$ defined by $[M^s] \mapsto [M^s]$ and $[M^a] \mapsto [M^a]$ on a set of free generators of $\mathrm{Gr}^{\mathrm{bi}}(\mathfrak{L})$ is a monomorphism of abelian groups which clearly is not always compatible with the multiplications as the domain is not associative (at least in the case that $\dim_{\mathbb{F}} \mathfrak{L}_{\mathrm{Lie}} < \infty$, see Proposition 5.10), but the codomain is associative.

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Skew braces with no proper left ideals

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Abstract. A skew brace $A = (A, \cdot, \circ)$ is said to be *left-simple* if $A \neq 1$ and it has no left ideal other than 1 and A. The purpose of this paper is to give a partial classification of the finite left-simple skew braces. A result of Stefanello and Trappeniers implies that finite left-simple skew braces correspond precisely to minimal Hopf–Galois structures on finite Galois extensions of fields.

Contents

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1 Introduction

A *skew brace* is a set $A = (A, \cdot, \circ)$ equipped with two group operations $\cdot$ and $\circ$ such that the so-called brace relation

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)$$

holds for all $a, b, c \in A$. For any $a \in A$, we write a^{-1} and $\overline{a}$, respectively, for the inverses of a in the groups $(A, \cdot)$ and $(A, \circ)$. We easily check that $(A, \cdot)$ and $(A, \circ)$ share the same identity 1. For each $a \in A$, it is straightforward to verify that the map

$$\lambda_a : A \longrightarrow A; \quad \lambda_a(x) = a^{-1} \cdot (a \circ x)$$

is an automorphism of $(A, \cdot)$, and it is also not hard to show that

$$\lambda : (A, \circ) \longrightarrow \mathrm{Aut}(A, \cdot); \quad a \longmapsto \lambda_a \tag{1}$$

is a group homomorphism. A *left ideal* of A is a subgroup I of $(A, \cdot)$ that is invariant under the action of λ, namely $\lambda_a(I) \subseteq I$ for all $a \in A$. Note that I is a subgroup of $(A, \circ)$ in this case. We introduce a new definition.

Definition 1.1. We shall say that a skew brace $A = (A, \cdot, \circ)$ is *left-simple* if $A \neq 1$ and it has no left ideal other than 1 and A.

Analogous to the Classification of Finite Simple Groups, it is natural to ask for a classification of the finite left-simple skew braces. The purpose of this paper is to give a partial classification.

Let $A = (A, \cdot, \circ)$ be any finite left-simple skew brace. The *additive group* of A, namely $(A, \cdot)$, must be characteristically simple because its characteristic subgroups are all left ideals of A. Hence, we have $(A, \cdot) \simeq T^n$ for some finite simple group T and $n \in \mathbb{N}$. We are able to prove a complete classification when T is abelian and when $n = 1$. We give a few fairly restrictive conditions that A must satisfy when T is non-abelian and $n \geq 2$.

Theorem 1.2. *Let $A = (A, \cdot, \circ)$ be a finite skew brace such that $(A, \cdot) \simeq C_p^n$ for some prime p and $n \in \mathbb{N}$. Then A is left-simple if and only if $n = 1$.*

Proof. We may regard the group homomorphism

$$\lambda : (A, \circ) \longrightarrow \mathrm{Aut}(A, \cdot) \simeq \mathrm{GL}_n(\mathbb{F}_p)$$

as a linear representation of the finite p-group $(A, \circ)$ over the field $\mathbb{F}_p$ that has characteristic p. The left ideals of the skew brace A then coincide with the λ-invariant subspaces of the vector space $(A, \cdot) \simeq \mathbb{F}_p^n$. This means that A is left-simple if and only if λ is irreducible. But it is well-known in modular representation theory (e.g. see [13, Proposition 6.2.1]) that λ is irreducible if and only if it is the trivial representation. This proves the theorem. □

The following corollary is immediate from Theorem 1.2.

Corollary 1.3. *Up to isomorphism, the finite left-simple skew braces with an abelian additive group are exactly* $\mathbb{F}_p = (\mathbb{F}_p, +, +)$, *where* p *is any prime and* $+$ *is the usual addition.*

Theorem 1.4. *Let* $A = (A, \cdot, \circ)$ *be a finite skew brace such that* $(A, \cdot) \simeq T^n$ *for some non-abelian simple group* T *and* $n \in \mathbb{N}$. *It is well-known that*

$$\mathrm{Aut}(T^n) = \mathrm{Aut}(T) \wr S_n = \mathrm{Aut}(T)^n \rtimes S_n,$$

where S_n *denotes the symmetric group on* n *letters.*

- *(a) In the case* $n = 1$, *we have* A *is left-simple if and only if* A *is an almost trivial skew brace, namely,* $a \circ b = b \cdot a$ *for all* $a, b \in A$.
- *(b) In the case* $n \geq 2$, *if* A *is left-simple, then* $(A, \circ)$ *is isomorphic to some subgroup of* $\mathrm{Aut}(T^n)$ *whose projection onto* S_n *is a transitive subgroup, and* $\mathrm{Im}(\lambda)$ *intersects trivially with* $\mathrm{Inn}(A, \cdot)$.

Remark 1.5. The motivation of this paper came from the study of minimal Hopf–Galois structures on finite Galois extensions. In the language of Hopf–Galois theory, our results above yield a partial classification of the minimal Hopf–Galois structures on finite Galois extensions (see Section 4).

Remark 1.6. A skew brace $A = (A, \cdot, \circ)$ is said to be *simple* if $A \neq 1$ and it has no ideal other than 1 and A. Recall that an *ideal* of A is a left ideal I of A that is normal in both $(A, \cdot)$ and $(A, \circ)$. In this case, we can naturally define a quotient skew brace structure on the set

$$A/I = \{a \cdot I : a \in A\} = \{a \circ I : a \in A\}.$$

Thus, ideals in a skew brace are a natural analog of normal subgroups in a group, and simple skew braces are arguably a closer analog of simple groups than left-simple skew braces. A classification of finite simple skew braces is desirable, but it seems to be out of reach at the moment. For example, in a simple skew brace $A = (A, \cdot, \circ)$, it is possible that $(A, \cdot)$ is abelian [1].

2 Preliminaries

Let $A = (A, \cdot, \circ)$ be a skew brace. For each $a \in A$, the maps

$$\begin{aligned} \lambda_a : A \longrightarrow A; &\quad \lambda_a(x) = a^{-1} \cdot (a \circ x) \\ \rho_a : A \longrightarrow A; &\quad \rho_a(x) = (a \circ x) \cdot a^{-1} \end{aligned}$$

are automorphisms of $(A, \cdot)$. Also, let

$$\mathrm{conj}(a) : A \longrightarrow A; \quad \mathrm{conj}(a)(x) = a \cdot x \cdot a^{-1}$$

denote the inner automorphism of $(A,\cdot)$ induced by a, so that

$$\rho_a = \mathrm{conj}(a)\lambda_a, \quad \lambda_a = \mathrm{conj}(a^{-1})\rho_a, \quad \mathrm{conj}(a) = \rho_a\lambda_a^{-1} \tag{2}$$

hold. It is well-known that

$$\begin{aligned} \lambda : (A,\circ) &\longrightarrow \mathrm{Aut}(A,\cdot); \quad a \longmapsto \lambda_a \\ \rho : (A,\circ) &\longrightarrow \mathrm{Aut}(A,\cdot); \quad a \longmapsto \rho_a \end{aligned}$$

are group homomorphisms. Also let

$$\mathrm{conj} : (A,\cdot) \longrightarrow \mathrm{Aut}(A,\cdot); \quad a \longmapsto \mathrm{conj}(a)$$

denote the natural group homomorphism.

For the homomorphisms λ and ρ, we consider the product

$$\Gamma := \mathrm{Im}(\lambda)\mathrm{Im}(\rho)$$

of their images. The next proposition is essentially [12, Proposition 2.1] and it implies that Γ is subgroup of $\mathrm{Aut}(A,\cdot)$ by the normality of $\mathrm{Inn}(A,\cdot)$.

Proposition 2.1. *We have the equalities*

$$\Gamma = \mathrm{Im}(\lambda)\mathrm{Inn}(A,\cdot) = \mathrm{Im}(\rho)\mathrm{Inn}(A,\cdot).$$

Proof. For any $a,b \in A$, a simple calculation using (2) yields that

$$\begin{aligned} \lambda_a \cdot \rho_b &= \lambda_a\mathrm{conj}(b)\lambda_b = \lambda_a\lambda_b \cdot \mathrm{conj}(\lambda_b^{-1}(b)), \\ \lambda_a \cdot \mathrm{conj}(b) &= \mathrm{conj}(a^{-1})\rho_a\mathrm{conj}(b) = \rho_a \cdot \mathrm{conj}(\rho_a^{-1}(a^{-1})b), \\ \rho_a \cdot \mathrm{conj}(b) &= \mathrm{conj}(\rho_a(b)^{-1})^{-1}\rho_a = \lambda_{\rho_a(b)^{-1}} \cdot \rho_{\rho_a(b)^{-1}}^{-1}\rho_a. \end{aligned}$$

The equalities now follow. ☐

Using the homomorphisms λ and ρ, there are two left ideals

$$J_1 := \mathrm{conj}^{-1}(\mathrm{Im}(\lambda)), \quad J_2 := \ker(\rho)$$

that we can construct. They are implicit in [2, Proof of Theorem A].

Proposition 2.2. *Both of J_1 and J_2 are left ideals of A.*

Proof. Clearly J_1 is a subgroup of $(A,\cdot)$. Let $j \in J_1$ and write $\mathrm{conj}(j) = \lambda_b$ for some $b \in A$. For any $a \in A$, it then follows that

$$\mathrm{conj}(\lambda_a(j)) = \lambda_a\mathrm{conj}(j)\lambda_a^{-1} = \lambda_a\lambda_b\lambda_a^{-1} \in \mathrm{Im}(\lambda),$$

which yields $\lambda_a(j) \in J_1$. This shows that J_1 is a left ideal of A.

Clearly J_2 is a subgroup of $(A,\circ)$. For any $a\in A$ and $j\in J_2$, we have

$$j\circ a \ = \rho_j(a)\cdot j = a \ \cdot j,$$

and so J_2 is also a subgroup of $(A,\cdot)$. The above further implies that

$$a\circ j\circ \overline{a} = a\circ(\overline{a}\cdot j) = (a\circ\overline{a})\cdot a^{-1}\cdot(a\circ j) = \lambda_a(j).$$

Since J_2 is in fact a normal subgroup of $(A,\circ)$, we obtain $\lambda_a(j)\in J_2$. This proves that J_2 is a left ideal of A. ☐

3 Proof of Theorem 1.4

In this section, let $A=(A,\cdot,\circ)$ be a finite skew brace with $(A,\cdot)\simeq T^n$, where T is a non-abelian simple group and $n\in\mathbb{N}$. Also, let

$$\Gamma := \mathrm{Im}(\lambda)\mathrm{Im}(\rho),\quad J_1 := \mathrm{conj}^{-1}(\mathrm{Im}(\lambda)),\quad J_2 := \ker(\rho)$$

be defined as in the previous section. By Proposition 2.1, we have

$$\mathrm{Inn}(A,\cdot)\le\Gamma\le\mathrm{Aut}(A,\cdot). \tag{3}$$

By Proposition 2.2, we know that J_1 and J_2 are left ideals of A.

3.1 The case when $n=1$

First, suppose that A is almost trivial. Then the left ideals of A are exactly the normal subgroups of $(A,\cdot)$. But $(A,\cdot)$ is non-abelian simple, so clearly A is left-simple.

Conversely, suppose that A is left-simple. Then A is non-trivial, namely, $a\circ b\neq a\cdot b$ for some $a,b\in A$, for otherwise any subgroup of $(A,\cdot)$ would be a left ideal of A. We suppose for contradiction that A is not almost trivial. Since A is neither trivial nor almost trivial, we obtain from [4, Theorem 1.1] that $(A,\circ)$ is not simple (the result of [4] is stated in terms of Hopf–Galois structures, but using [11, Theorem 3.1], for example, to translate it into the language of skew braces, it says that a finite skew brace with a non-abelian simple multiplicative group is either trivial or almost trivial). We then have $(A,\circ)\not\simeq(A,\cdot)$. By order consideration, it follows that

$$\mathrm{Im}(\lambda)\not\supseteq\mathrm{Inn}(A,\cdot)\quad\text{and}\quad\mathrm{Im}(\rho)\not\supseteq\mathrm{Inn}(A,\cdot). \tag{4}$$

Indeed, note that $\mathrm{Inn}(A,\cdot)\simeq(A,\cdot)$ has order $|A|$, while

$$\mathrm{Im}(\lambda)\simeq(A,\circ)/\ker(\lambda)\quad\text{and}\quad\mathrm{Im}(\rho)\simeq(A,\circ)/\ker(\rho)$$

have order at most $|A|$. This implies that

$$\begin{cases}(A,\circ)\simeq\mathrm{Im}(\lambda)=\mathrm{Inn}(A,\cdot)\simeq(A,\cdot) & \text{if } \mathrm{Im}(\lambda)\supseteq\mathrm{Inn}(A,\cdot),\\ (A,\circ)\simeq\mathrm{Im}(\rho)=\mathrm{Inn}(A,\cdot)\simeq(A,\cdot) & \text{if } \mathrm{Im}(\rho)\supseteq\mathrm{Inn}(A,\cdot),\end{cases}$$

and we have a contradiction in both cases.

Observe that $J_1 \neq A$ because $\mathrm{Im}(\lambda) \not\supseteq \mathrm{Inn}(A,\cdot)$, and $J_2 \neq A$ because A is not almost trivial. Since J_1, J_2 are left ideals of A by Proposition 2.2, the left-simplicity of A implies that $J_1 = J_2 = 1$.

Note that Γ is an almost simple group with socle $(A,\cdot) \simeq T$ by (3). We consider its factorization $\Gamma = \mathrm{Im}(\lambda)\mathrm{Im}(\rho)$. By (4), both of the factors are core-free, in the sense that they do not contain the socle. Moreover:

- Since $J_1 = 1$, the factor $\mathrm{Im}(\lambda)$ intersects trivially with $\mathrm{Inn}(A,\cdot)$ and so it embeds into $\mathrm{Out}(A,\cdot)$. Since $\mathrm{Out}(A,\cdot)$ is solvable by Schreier Conjecture (a consequence of the Classification of Finite Simple Groups), we deduce that $\mathrm{Im}(\lambda)$ is also solvable. Also, $\mathrm{Im}(\lambda) \neq 1$ because A is non-trivial.

- Since $J_2 = 1$, the factor $\mathrm{Im}(\rho)$ is isomorphic to $(A,\circ)$.

Fortunately, the core-free factorizations of finite almost simple groups with a solvable factor have been classified in [9, Theorem 1.1]. Put $H := \mathrm{Im}(\lambda)$, which is solvable, and $K := \mathrm{Im}(\rho)$. Identifying $(A,\cdot)$ with T, we have:

(1) $|H| \neq 1,\ |H \cap \mathrm{Inn}(T)| = 1$;

(2) $|H|$ divides $|\mathrm{Out}(T)|$;

(3) $|K| = |T|$.

From Proposition 2.1, we also know that

$$\frac{K}{K \cap \mathrm{Inn}(T)} \simeq \frac{K\mathrm{Inn}(T)}{\mathrm{Inn}(T)} = \frac{H\mathrm{Inn}(T)}{\mathrm{Inn}(T)} \simeq \frac{H}{H \cap \mathrm{Inn}(T)}.$$

Together with (1) and (3), this yields the equality:

(4) $|T| = |H||K \cap \mathrm{Inn}(T)|$.

We now use [9] to show that all candidates of (H, K) are impossible.

Remark 3.1. In the following tables, the orders of certain groups are listed. For the linear groups, they can be constructed in Magma via

- `PSL(n,q)` for the projective special linear group on $\mathrm{GF}(q)^n$;
- `PSU(n,q)` for the projective special unitary group on $\mathrm{GF}(q^2)^n$;
- `PSp(2n,q)` for the projective sympletic group on $\mathrm{GF}(q)^{2n}$;
- `PSO(2n+1,q)` for the projective special orthogonal group on $\mathrm{GF}(q)^{2n+1}$ (the symbol $\mathrm{P}\Omega_{2n+1}(q)$ denotes its derived subgroup);

- `PSOPlus(2n,q)` and `PSOMinus(2n,q)`, respectively, for the projective special orthogonal groups on $\mathrm{GF}(q)^{2n}$ with Witt indices n and $n-1$ (the symbols $\mathrm{P}\Omega_{2n}^{+}(q)$ and $\mathrm{P}\Omega_{2n}^{-}(q)$ denote their derived subgroups);
- `AGammaL(n,q)` for the affine semilinear group on $\mathrm{GF}(q)^n$;
- `ChevalleyGroup("G",2,q)` for the Chevalley group or exceptional group of type G_2 over the field $\mathrm{GF}(q)$.

For the non-projective linear groups, simply remove `P` in the above. For the Mathieu groups, they can be constructed in `GAP` via

- `MathieuGroup(n)` for the Mathieu group acting on n objects.

The orders of these groups may then be computed. The orders of the outer automorphism groups $\mathrm{Out}(T)$ may be computed via

$$|\mathrm{Out}(T)| = |\mathrm{Aut}(T)|/|\mathrm{Inn}(T)| = |\mathrm{Aut}(T)|/|T|.$$

For the symmetric and alternating groups, their orders are obvious, so we do not list them explicitly.

Suppose first that K is also solvable. By [9, Proposition 4.1], for

$$(H', K') = (H, K) \quad \text{or} \quad (H', K') = (K, H),$$

the next two tables give all the possibilities.

T	$\lvert K' \cap \mathrm{Inn}(T)\rvert$ is divisible by	$\lvert K' \cap \mathrm{Inn}(T)\rvert$ divides
$\mathrm{PSL}_2(q)$	q	$\frac{q(q-1)}{(2,q-1)}$

T	$\lvert\mathrm{Out}(T)\rvert$	$\lvert H'\rvert$ is divisible by	$\lvert K'\rvert$ is divisible by
$\mathrm{PSL}_2(7)$	2	7	$\lvert\mathrm{S}_4\rvert$
$\mathrm{PSL}_2(11)$	2	$11 \cdot 5$	$\lvert\mathrm{A}_4\rvert$
$\mathrm{PSL}_2(23)$	2	$23 \cdot 11$	$\lvert\mathrm{S}_4\rvert$
$\mathrm{PSL}_3(3)$	2	13	$3^2 \cdot 2 \cdot \lvert\mathrm{S}_4\rvert$
$\mathrm{PSL}_3(3)$	2	$13 \cdot 3$	$\lvert\mathrm{A\Gamma L}_1(9)\rvert$
$\mathrm{PSL}_3(4)$	12	$7 \cdot 3 \cdot \lvert\mathrm{S}_3\rvert$	$2^4 \cdot 3 \cdot \lvert\mathrm{D}_{10}\rvert \cdot 2$
$\mathrm{PSL}_3(8)$	6	$73 \cdot 9$	$2^{3+6} \cdot 7^2 \cdot 3$
$\mathrm{PSU}_3(8)$	18	$57 \cdot 9$	$2^{3+6} \cdot 63 \cdot 3$
$\mathrm{PSU}_4(2)$	2	$2^4 \cdot 5$	$3^{1+2} \cdot 2 \cdot \lvert\mathrm{A}_4\rvert$
$\mathrm{PSU}_4(2)$	2	$2^4 \cdot \lvert\mathrm{D}_{10}\rvert$	$3^{1+2} \cdot 2 \cdot \lvert\mathrm{A}_4\rvert$
$\mathrm{PSU}_4(2)$	2	$2^4 \cdot 5 \cdot 4$	$3^3 \cdot \lvert\mathrm{S}_3\rvert$

M_{11}	1	$11\cdot 5$	$\lvert\mathrm{M}_9\rvert\cdot 2$

In the second table, observe that

$$|\mathrm{A\Gamma L}_1(9)| = 144, \quad |\mathrm{D}_{10}| = 10, \quad |\mathrm{M}_9| = 72.$$

We see that neither $|H'|$ nor $|K'|$ divides $|\mathrm{Out}(T)|$, which contradicts (2).

In the first table, we must have $(H',K') = (H,K)$ by (1). Note that

$$|T| = |\mathrm{PSL}_2(q)| = \frac{q(q-1)(q+1)}{(2,q-1)},$$
$$|\mathrm{Out}(T)| = |\mathrm{Out}(\mathrm{PSL}_2(q))| = (2,q-1)f,$$

where $q = p^f$ and p is a prime. We then see from (2) and the third column of the table that $|H||K\cap \mathrm{Inn}(T)|$ has to divide

$$(2,q-1)f\cdot\frac{q(q-1)}{(2,q-1)} = q(q-1)f,$$

which is strictly smaller than $|T|$ because

$$f < \frac{2^f+1}{2} \leq \frac{q+1}{(2,q-1)}.$$

This yields a contradiction to (4).

We have shown that K must be insolvable. From here, we argue depending on the type of T. Let us remark that T cannot be an exceptional group of Lie type by [9, Proposition 4.2].

I. T is an alternating group

The possibilities for (H,K) are given in [9, Proposition 4.3 (a)∼(f)]. Let $T = A_m$, where $m\geq 5$, and put $\Omega_m = \{1,2,\dots,m\}$. For (a), it says H is transitive on Ω_m, and so $|H|\geq m$. For (b), it says H is 2-homogeneous on Ω_m, namely, H is transitive on the 2-subsets of Ω_m, and so $|H|\geq \frac{m(m-1)}{2}$. For (c) and (f), there is a list of candidates for H, and we see that $|H|\geq 5$. Since $|\mathrm{Out}(T)|\in\{2,4\}$, we obtain a contradiction to (2) in these cases. For both (d) and (e), it states $T = A_6$ and $K\in\{\mathrm{PSL}_2(5),\mathrm{PGL}_2(5)\}$. But then this yields $|T| = 360$ while $|K|\in\{60,120\}$, and this contradicts (3).

II. T is a sporadic group

The possibilities for (H,K) are given in [9, Proposition 4.4 (a)∼(c)]. For (a) and (b), it says $(T,K) = (\mathrm{M}_{12},\mathrm{M}_{11}), (\mathrm{M}_{24},\mathrm{M}_{23})$, respectively, which contradicts (3). For (c), a list of candidates for H is given, and we find that $|H|\geq 3$. Since $|\mathrm{Out}(T)|\in\{1,2\}$, this contradicts (2).

III. T is a classical group of Lie type and is not an alternating group

The possibilities for (H, K) are given in [9, Tables 1.1 and 1.2]. However, they only give an upper bound for $H \cap \mathrm{Inn}(T)$, which is not enough for us. For [9, Table 1.1], we shall further apply [3, Theorem 1], which provides a lower bound for $|H|$ and we can derive a contradiction to (2). For [9, Table 1.2], we shall use the description of $K \cap \mathrm{Inn}(T)$ and obtain a contradiction to (4). We summarize all the possibilities in the two tables below.

	T	$\lvert H\rvert$ is divisible by
(a)	$\mathrm{PSL}_n(q)$, $n \geq 2$	$\frac{q^n-1}{q-1}$
(b)	$\mathrm{PSL}_4(q)$	$\frac{q^3(q^3-1)}{(2,q-1)}$
(c)	$\mathrm{PSp}_{2m}(q)$, $m \geq 2$, q even	$q^m(q^m-1)$
(d)	$\mathrm{PSp}_4(q)$, q even	$q^2(q^2-1)$
(e)	$\mathrm{PSp}_4(q)$, q odd	$q^3(q^2-1)$
(f)	$\mathrm{PSU}_{2m}(q)$, $m \geq 2$	$\frac{q^{2m}(q^{2m}-1)}{q+1}$
(g)	$\Omega_{2m+1}(q)$, $m \geq 3$, q odd	$\frac{1}{2}q^{m(m+1)/2}(q^m-1)$
(h)	$\mathrm{P}\Omega^+_{2m}(q)$, $m \geq 5$	$\frac{q^m(q^m-1)}{(2,q-1)}$
(i)	$\mathrm{P}\Omega^+_8(q)$	$\frac{q^4(q^4-1)}{(2,q-1)}$

T	$\lvert T\rvert$	$\lvert\mathrm{Out}(T)\rvert$	$\lvert K \cap \mathrm{Inn}(T)\rvert$ divides
$\mathrm{PSL}_2(11)$	660	2	$\lvert \mathrm{A}_5\rvert$
$\mathrm{PSL}_2(16)$	4080	4	$\lvert \mathrm{A}_5\rvert$
$\mathrm{PSL}_2(19)$	3420	2	$\lvert \mathrm{A}_5\rvert$
$\mathrm{PSL}_2(29)$	12180	2	$\lvert \mathrm{A}_5\rvert$
$\mathrm{PSL}_2(59)$	102660	2	$\lvert \mathrm{A}_5\rvert$
$\mathrm{PSL}_4(3)$	6065280	4	$3^3 \cdot \lvert\mathrm{PSL}_3(3)\rvert$
$\mathrm{PSL}_4(3)$	6065280	4	$4 \cdot \lvert\mathrm{PSL}_2(9)\rvert \cdot 2$
$\mathrm{PSL}_4(4)$	987033600	4	$5 \cdot \lvert\mathrm{PSL}_2(16)\rvert \cdot 2$
$\mathrm{PSL}_5(2)$	9999360	2	$2^6 \cdot \lvert\mathrm{S}_3\rvert \cdot \lvert\mathrm{PSL}_3(2)\rvert$
$\mathrm{PSp}_4(3)$	25920	2	$2^4 \cdot \lvert\mathrm{A}_5\rvert$
$\mathrm{PSp}_4(3)$	25920	2	$2^4 \cdot \lvert\mathrm{S}_6\rvert$
$\mathrm{PSp}_4(5)$	4680000	2	$\lvert\mathrm{PSL}_2(5^2)\rvert \cdot 2$
$\mathrm{PSp}_4(7)$	138297600	2	$\lvert\mathrm{PSL}_2(7^2)\rvert \cdot 2$
$\mathrm{PSp}_4(11)$	12860654400	2	$\lvert\mathrm{PSL}_2(11^2)\rvert \cdot 2$
$\mathrm{PSp}_4(23)$	20674026236160	2	$\lvert\mathrm{PSL}_2(23^2)\rvert \cdot 2$
$\mathrm{Sp}_6(2)$	1451520	1	$\lvert\mathrm{S}_8\rvert$
$\mathrm{PSp}_6(3)$	4585351680	2	$\lvert\mathrm{PSL}_2(27)\rvert \cdot 3$
$\mathrm{PSU}_3(3)$	6048	2	$\lvert\mathrm{PSL}_2(7)\rvert$
$\mathrm{PSU}_3(5)$	126000	6	$\lvert\mathrm{A}_7\rvert$
$\mathrm{PSU}_4(3)$	3265920	8	$\lvert\mathrm{PSL}_3(4)\rvert$
$\mathrm{PSU}_4(8)$	34693789777920	6	$2^{12} \cdot \lvert\mathrm{SL}_2(64)\rvert \cdot 7$

$\Omega_7(3)$	4585351680	2	$\lvert G_2(3)\rvert$
$\Omega_7(3)$	4585351680	2	$\lvert \mathrm{Sp}_6(2)\rvert$
$\Omega_9(3)$	65784756654489600	2	$\lvert \Omega_8^-(3)\rvert \cdot 2$
$\Omega_8^+(2)$	174182400	6	$\lvert \mathrm{Sp}_6(2)\rvert$
$\Omega_8^+(2)$	174182400	6	$\lvert A_9\rvert$
$\mathrm{P}\Omega_8^+(3)$	4952179814400	24	$\lvert \Omega_7(3)\rvert$
$\mathrm{P}\Omega_8^+(3)$	4952179814400	24	$\lvert \Omega_8^+(2)\rvert$

In the second table, let $\mathcal{K}$ be the number in the last column. We deduce from (2) that $|H||K \cap \mathrm{Inn}(T)|$ has to divide $|\mathrm{Out}(T)|\mathcal{K}$. Note that:

$\lvert \mathrm{PSL}_3(3)\rvert$	5616	$\lvert \mathrm{PSL}_2(7)\rvert$	168
$\lvert \mathrm{PSL}_2(9)\rvert$	360	$\lvert \mathrm{PSL}_3(4)\rvert$	20160
$\lvert \mathrm{PSL}_2(16)\rvert$	4080	$\lvert \mathrm{SL}_2(64)\rvert$	262080
$\lvert \mathrm{PSL}_3(2)\rvert$	168	$\lvert G_2(3)\rvert$	4245696
$\lvert \mathrm{PSL}_2(5^2)\rvert$	7800	$\lvert \mathrm{Sp}_6(2)\rvert$	1451520
$\lvert \mathrm{PSL}_2(7^2)\rvert$	58800	$\lvert \Omega_8^-(3)\rvert$	10151968619520
$\lvert \mathrm{PSL}_2(11^2)\rvert$	885720	$\lvert \Omega_7(3)\rvert$	4585351680
$\lvert \mathrm{PSL}_2(23^2)\rvert$	74017680	$\lvert \Omega_8^+(2)\rvert$	174182400
$\lvert \mathrm{PSL}_2(27)\rvert$	9828		

It is then straightforward to check that

$$|\mathrm{Out}(T)|\mathcal{K} < |T|$$

in every row of the second table. This yields a contradiction to (4).

In the first table, write $q = p^f$, where p is a prime. As listed in [9, Table 2.1], for example, we know that:

T	$\lvert\mathrm{Out}(T)\rvert$
$\mathrm{PSL}_2(q)$	$(2, q-1) \cdot f$
$\mathrm{PSL}_n(q)$, $n \geq 3$	$(n, q-1) \cdot 2 \cdot f$
$\mathrm{PSU}_n(q)$, $n \geq 3$	$(n, q+1) \cdot 2f$
$\mathrm{PSp}_{2m}(q)$, $(m,p) \neq (2,2)$	$(2, q-1) \cdot f$
$\mathrm{PSp}_4(q)$, q even	$2f$
$\Omega_{2m+1}(q)$, $m \geq 3$, q odd	$2 \cdot f$
$\mathrm{P}\Omega_{2m}^+(q)$, $m \geq 5$, $q^m \not\equiv 1 \pmod 4$	$2 \cdot (2, q-1) \cdot f$
$\mathrm{P}\Omega_{2m}^+(q)$, $m \geq 5$, $q^m \equiv 1 \pmod 4$	$8 \cdot f$
$\mathrm{P}\Omega_8^+(q)$	$(2 + (2, q-1))! \cdot f$

For an integer z, write $v_p(z)$ for the p-adic valuation of z. Using the trivial bound that $v_p(2) \leq 1$ when 2 divides $|\mathrm{Out}(T)|$, we see that:

	$v_p(\lvert\mathrm{Out}(T)\rvert)$ is at most	$v_p(\lvert H\rvert)$ is at least
(b)	$1+v_p(f)$	$3f$
(c)	$1+v_p(f)$, $p=2$	mf, $m\geq 2$
(d)	$1+v_p(f)$, $p=2$	$2f$
(e)	$v_p(f)$, $p\geq 3$	$3f$
(f)	$1+v_p(f)$	$2mf$, $m\geq 2$
(g)	$v_p(f)$, $p\geq 3$	$\frac{m(m+1)}{2}f$, $m\geq 3$
(h)	$3+v_p(f)$	mf, $m\geq 5$
(i)	$3+v_p(f)$	$4f$

Since $v_p(f)\leq \log_p(f)<f$ and $1\leq f$, in cases (b)∼(i), we deduce that

$$v_p(|\mathrm{Out}(T)|)<v_p(|H|).$$

But then $|H|$ cannot divide $|\mathrm{Out}(T)|$, and this contradicts (2).

Finally, we deal with case (a). If $n=2$, then

$$|\mathrm{Out}(T)|\leq 2f<2^f+1\leq p^f+1=\frac{q^2-1}{q-1}\leq |H|,$$

which contradicts (2). If $n\geq 3$ with $(fn,p)=(6,2)$, then

$$\begin{cases}|\mathrm{Out}(T)|=2<\frac{2^6-1}{2-1}\leq |H| & \text{for } (f,n)=(1,6),\ q=2,\\ |\mathrm{Out}(T)|=12<\frac{4^3-1}{4-1}\leq |H| & \text{for } (f,n)=(2,3),\ q=4,\end{cases}$$

which again contradicts (2). If $n\geq 3$ with $(fn,p)\neq(6,2)$, then we deduce from Zsigmondy's theorem [14] (also see [10, Theorem 3]) that $p^{fn}-1$ has a primitive prime divisor ℓ. In other words, the prime ℓ is such that

$$\ell\mid p^{fn}-1,\quad \ell\nmid p^k-1 \text{ for all } 1\leq k\leq fn-1. \tag{5}$$

It follows that ℓ divides $\frac{q^n-1}{q-1}$ and hence $|H|$. By (2), we see that ℓ divides

$$|\mathrm{Out}(T)|=(n,q-1)\cdot 2\cdot f.$$

Since $\ell\nmid q-1$ and clearly $\ell\neq 2$, this means that $\ell\mid f$. But $p\equiv p^\ell \pmod{\ell}$ so we deduce from the first condition in (5) that

$$p^{\frac{f}{\ell}n}-1\equiv p^{fn}-1\equiv 0 \pmod{\ell}.$$

This contradicts the second condition in (5).

We have now ruled out every possibility. Therefore, the assumption that A is not almost trivial is false. This completes the proof of Theorem 1.4(a).

3.2 The case when $n \geq 2$.

Suppose that A is left-simple. Note that

$$\lambda_a \equiv \rho_a \pmod{\mathrm{Inn}(A,\cdot)} \tag{6}$$

for all $a \in A$ by (2), and that

$$\mathrm{Im}(\lambda) \not\subseteq \mathrm{Inn}(A,\cdot), \tag{7}$$

for otherwise $T \times 1 \times \cdots \times 1$ would be a non-trivial proper left ideal of A. We then see that $J_1 \neq A$ by order consideration. Indeed, if $J_1 = A$, namely the inclusion $\mathrm{Inn}(A,\cdot) \subseteq \mathrm{Im}(\lambda)$ holds, then we must in fact have an equality because $\mathrm{Inn}(A,\cdot) \simeq (A,\cdot)$ has order $|A|$, while

$$\mathrm{Im}(\lambda) \simeq (A,\circ)/\ker(\lambda)$$

has order at most $|A|$. But this contradicts (7). From (6) and (7), we also obtain $J_2 \neq A$. Since J_1, J_2 are left ideals of A by Proposition 2.2, the left-simplicity of A implies that $J_1 = J_2 = 1$.

Since $J_1 = 1$, the image $\mathrm{Im}(\lambda)$ of λ intersects trivially with $\mathrm{Inn}(A,\cdot)$, as claimed in the theorem.

Since $J_2 = 1$, the multiplicative group $(A,\circ)$ embeds into

$$\mathrm{Aut}(A,\cdot) \simeq \mathrm{Aut}(T)^n \rtimes S_n$$

via ρ, and let P denote the projection of $\mathrm{Im}(\rho)$ onto S_n. Since

$$\mathrm{Inn}(A,\cdot) \simeq \mathrm{Inn}(T)^n \subseteq \mathrm{Aut}(T)^n,$$

the projection of $\mathrm{Im}(\lambda)$ onto S_n is also equal to P by (6). For any orbit O of the natural action of P on $\{1,2,\dots,n\}$, the subgroup of $(A,\cdot)$ given by

$$\prod_{i\in O} T_i,$$

where T_i denotes the ith copy of T in $(A,\cdot) \simeq T^n$, is then a left ideal of A. But A is left-simple, so necessarily $O = \{1,2,\dots,n\}$. This proves that P is a transitive subgroup of S_n.

This completes the proof of Theorem 1.4(b).

Remark 3.2. The condition that $(A,\circ)$ is isomorphic to a subgroup, X say, of $\mathrm{Aut}(T^n)$ whose projection onto S_n is transitive is very restrictive. To see why, consider the natural homomorphism

$$\phi : X \longhookrightarrow \mathrm{Aut}(T)^n \rtimes S_n \longrightarrow \mathrm{Out}(T)^n \rtimes S_n.$$

Its kernel $\ker(\phi)$ lies in $\mathrm{Inn}(T)^n \simeq T^n$, and its image $\mathrm{Im}(\phi) \simeq X/\ker(\phi)$ has the same projection onto S_n as X, which is transitive. It follows that

$$\exists R \leq T^n \text{ s.t. } n \mid [T^n : R] \text{ and } [T^n : R] \mid |\mathrm{Out}(T)|^n \cdot n!. \tag{8}$$

In particular, the prime factors of n must all divide $|T|$.

For example, take $T = (T, \cdot) = (A_5, \cdot)$. Note that

$$|A_5| = 60 = 2^2 \cdot 3 \cdot 5 \quad \text{and} \quad |\mathrm{Out}(A_5)| = 2.$$

We see that $(A_5^n, \cdot, \circ)$ can be left-simple only when $n = 2^x 3^y 5^z$, where $x, y, z \in \mathbb{N}_{\geq 0}$. Moreover, a skew brace $(A_5^2, \cdot, \circ)$ cannot be left-simple because A_5^2 has no subgroup of index $2, 4$, or 8.

4 Connection with minimal Hopf–Galois structures

Let L/K be a finite extension of fields. A *Hopf–Galois structure* on L/K is a finite-dimensional cocommutative Hopf K-algebra $\mathcal{H}$ that acts on L via a K-algebra homomorphism $\mathcal{H} \longrightarrow \mathrm{End}_K(L)$ such that the action respects the counit ϵ, multiplication m, and comultiplication Δ, in the sense that

$$h(1) = \epsilon(h)(1), \quad h(xy) = m(\Delta(h)(x \otimes y))$$

for all $h \in \mathcal{H}$, $x, y \in L$, and the natural K-homomorphism

$$L \otimes_K \mathcal{H} \longrightarrow \mathrm{End}_K(L)$$

is bijective. The notion of a Hopf–Galois structure was introduced by [5] (or see [6] for detailed overview). For each Hopf K-subalgebra $\mathcal{H}'$ of $\mathcal{H}$, let

$$L^{\mathcal{H}'} = \{x \in L : h'(x) = \epsilon(h')x \text{ for all } h' \in \mathcal{H}'\}$$

denote its fixed field. This yields a Hopf–Galois correspondence

$$\begin{aligned} \Phi_{\mathcal{H}} : \{\text{Hopf } K\text{-subalgebras of } \mathcal{H}\} &\longrightarrow \{\text{intermediate fields of } L/K\}; \\ \mathcal{H}' &\longmapsto L^{\mathcal{H}'} \end{aligned}$$

that is analogous to the Galois correspondence in Galois theory. Indeed, in the case that L/K is a G-Galois extension, the group ring $K[G]$ is naturally a Hopf–Galois structure on L/K, and the map $\Phi_{K[G]}$ above reduces to the usual Galois correspondence

$$\begin{aligned} \{\text{subgroups of } G\} &\longrightarrow \{\text{intermediate fields of } L/K\}; \\ G' &\longmapsto L^{G'} \end{aligned}$$

because the Hopf K-subalgebras of $K[G]$ are exactly the group rings $K[G']$, where G' ranges over the subgroups of G.

It is known by [5] that the map $\Phi_{\mathcal{H}}$ is always injective, but it need not be surjective in general. We can of course ask when $\Phi_{\mathcal{H}}$ is surjective. Here, we consider the other extreme and ask when $\Phi_{\mathcal{H}}$ is as far from being surjective as possible, i.e. when $\mathrm{Im}(\Phi_{\mathcal{H}})$ only contains L and K. Since $\Phi_{\mathcal{H}}$ is injective, this is precisely the case when K and $\mathcal{H}$ are the only Hopf K-subalgebras of $\mathcal{H}$. The next definition is due to [7].

Definition 4.1. A Hopf–Galois structure $\mathcal{H}$ on L/K is said to be *minimal* if $\dim_K(\mathcal{H}) \neq 1$ and $\mathcal{H}$ has no Hopf K-subalgebras other than K and $\mathcal{H}$.

The theory of Hopf–Galois structures applies to all extensions. But when restricted to separable extensions, we have a very nice group-theoretic classification of Hopf–Galois structures thanks to [8]. In addition, when further restricted to Galois extensions, the classification may be translated into the language of skew braces. The next theorem is from [11, Theorem 3.1].

Theorem 4.2. *Let L/K be any finite Galois extension of fields with Galois group $(G, \circ)$. There is a bijective correspondence between:*

(1) *the Hopf–Galois structures $\mathcal{H}$ on L/K;*

(2) *the binary operations $\cdot$ on G such that $(G, \cdot, \circ)$ is a skew brace.*

Specifically, such an operation $\cdot$ is associated to the Hopf–Galois structure

$$L[(G,\cdot)]^{(G,\circ)} = \left\{ \sum_{\tau\in G} \ell_\tau \tau \in L[(G,\cdot)] : \sigma(\ell_\tau) = \ell_{\lambda_\sigma(\tau)} \text{ for all } \sigma,\tau \in G \right\}$$

whose action on L is defined by

$$\left(\sum_{\tau\in G} \ell_\tau \tau \right)(x) = \sum_{\tau\in G} \ell_\tau \tau(x)$$

for all $x \in L$, where λ is the lambda map (1) of the skew brace $(G, \cdot, \circ)$.

Let us remark that the Hopf–Galois structure corresponding to the trivial skew brace is the group ring $K[G]$ and is called the *classical structure*, while the Hopf–Galois structure corresponding to the almost trivial skew brace is called the *canonical non-classical structure*.

The next theorem is from [11, Corollary 4.1].

Theorem 4.3. *Let L/K be any finite Galois extension of fields with Galois group $(G, \circ)$. Let $\cdot$ be any binary operation on G such that $(G, \cdot, \circ)$ is a skew brace and let $\mathcal{H}$ be the associated Hopf–Galois structure on L/K. For any subgroup I of G, the following are equivalent:*

(1) *the intermediate field L^I lies in the image of $\Phi_{\mathcal{H}}$;*

(2) *the subgroup I is a left ideal of $(G, \cdot, \circ)$.*

In particular, the following are equivalent:

(*a*) *the Hopf–Galois structure $\mathcal{H}$ is minimal;*

(*b*) *the skew brace* $(G, \cdot, \circ)$ *is left-simple.*

Therefore, in the language of Hopf–Galois theory, our Theorem 1.2, Corollary 1.3, and Theorem 1.4 give a partial classification of the minimal Hopf–Galois structures on finite Galois extensions of fields.

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